

Picture languages: Tiling Systems versus Tile Rewriting Grammars

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Abstract

Two formal models of pictures, i.e., 2D languages are compared: Tiling Systems and Tile Rewriting Grammars, which resp. extend to 2D the Regular and Context-Free languages. Two results extending classical language properties into 2D are proved. First, non-recursive TRG coincide with TS. Second, non-self-embedding TRG are suitably defined as corner grammars, showing that they generate TS languages. The proofs exploit newly introduced language substitutions, also nested and iterated.

Key words: picture language, picture grammar, 2D language, tiling system, recognizable picture language, tile rewriting grammar, 2D regular expression, autoinclusive derivation

1 Introduction

Since digital pictures can be thought as two dimensional texts, methods based on formal language theory have been considered since long time [9] for defining and processing images too. In order to define a set of rectangular pictures, i.e., a 2D

* Work partially supported by MIUR, Progetto *Linguaggi formali e automi, teoria e applicazioni*.

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language, one can then use the classical methods of automata, grammars, and homomorphic characterizations (see e.g. [6], [11], [7]). The corresponding formal models give rise to 2D language families, which in some cases nicely extend the basic properties of corresponding string language families. The two language families we are concerned with are 2D generalizations of the regular and context-free 1D cases.

The Tiling Systems *TS* [4] have attracted much interest because of their elegant characterization as the projection of locally testable languages, and of the preservation of several properties of regular languages. Our work introduces and systematically applies the notion of language substitution in 2D. By partitioning a picture into homogeneous subpictures, substitutions can operate in 2D without tearing the picture. A so-called block picture models the structure of an image made by juxtaposed uniform fields (called blocks), relying on a closure operation due to Simplot [10]. Block substitution is an operation conceptually similar to the collage operation of [2]. It is shown that TS languages are closed by block substitution. Greibach's [5] notion of nested iterated substitution is also extended and adapted to a special case for TS languages.

Concerning 2D context-free generalizations, we study the recent Tile Rewriting Grammar *TRG* [8] model, which conceptually stemmed from TS. Such grammars feature so called isometric [11] rewriting rules, which replace a subarray with another one of the same size. The nested application of block substitution offers an alternative crisper exposition of the original picture derivation mechanism of TRG grammars.

The relationships between TRG and TS languages families, beyond strict inclusion, were essentially unknown and are the subject of this presentation. Two main properties are proved, which reinforce the qualification of TS and TRG as good generalizations of regular and context-free string languages.

First, we show that non-recursive TRG grammars are equivalent to TS. In 1D the statement becomes: non recursive context-free grammars with regular expressions in the right parts of the rules¹ are equivalent to regular languages. The property descends immediately from the closure of regular languages with respect to language substitution.

Second, it is known [3] that a context-free grammar without self-embedding derivations defines a regular language. It was not obvious how to reformulate the non-self-embedding condition in 2D, in such a way that the TRG would generate a TS language. We propose a new 2D analogous of non-self-embedding grammars, namely corner grammars, showing that corner grammars, even if recursive, generate TS languages.

¹ Also named extended BNF or regular right part grammars.

Sect. 2 presents the basic definitions, together with TS. Sect. 3 presents different concepts of 2D substitutions and proves some closure properties of TS languages. Sect. 4 introduces TRGs and finally compares them with TS.

2 Basic Definitions

We briefly recall a few standard definitions. The reader may consult [4] for more detailed and formal definitions. A *picture* on a finite alphabet Σ is a two-dimensional rectangular array of elements in Σ . The *size* $|p|$ of a picture p is the pair $(|p|_{row}, |p|_{col})$ of its number of rows and columns. A *pixel* $p(i, j)$, $1 \leq i \leq |p|_{row}$, $1 \leq j \leq |p|_{col}$, is the element at position (i, j) in the array p . The indices grow from top to bottom for the rows and from left to right for the columns.

Let $\Sigma^{+,+}$ be the set of all nonempty pictures over Σ , and $\Sigma^{*,*}$ be $\Sigma^{+,+} \cup \{\lambda\}$, where λ is the empty picture. For $h, k \geq 1$, $\Sigma^{h,k}$ (resp. $\Sigma^{h,*}$, $\Sigma^{*,k}$) is the set of all pictures of size (h, k) (resp. with h rows, with k columns). A picture language over Σ is a subset of $\Sigma^{*,*}$. If all pixels of a picture p over Σ belong to an alphabet $\Sigma' \subseteq \Sigma$, p is called Σ' -*homogeneous*. A picture $\{a\}$ -homogeneous for some $a \in \Sigma$ is called an a -*picture*, or also a *homogeneous* picture. If $a \in \Sigma$, $a^{h,k}$ stands for the a -picture in $\Sigma^{h,k}$, while $a^{+,+}$ stands for the set of a -pictures in $\Sigma^{+,+}$.

Notation: The letters p, q usually stand for pictures. The pairs of letters (i, j) , $(i_1, j_1) \dots$ are typically used to denote the coordinates (or *position*) of a pixel in a picture (e.g., $1 \leq i \leq |p|_{row}$, $1 \leq j \leq |p|_{col}$). The upper case Greek letters $\Sigma, \Gamma, \Delta, \Xi$ are finite alphabets. The lower case letters a, b, c, d and (when useful) upper case letters A, B, C, D, X denote symbols of an alphabet. A singleton $\{p\}$ is denoted, when no confusion can arise, by p itself.

We shortly present, out of the many picture-combining and transforming operators, those needed in the remainder. The projection by mapping $\pi : \Sigma \rightarrow \Delta$ of a picture $p \in \Sigma^{+,+}$ is a picture $p' \in \Delta^{+,+}$ such that $|p| = |p'|$ and $p'(i, j) = \pi(p(i, j))$ for every position (i, j) of p . Projections can be extended to languages as usual. The (clockwise) *rotation* of a picture p , $rot(p)$, is described as follows:

$$\begin{array}{cccc}
 p(1, 1) & \dots & p(1, |p|_{col}) & \\
 \vdots & \ddots & \vdots & \\
 p(|p|_{row}, 1) & \dots & p(|p|_{row}, |p|_{col}) & \\
 \end{array}
 \qquad
 \begin{array}{cccc}
 p(|p|_{row}, 1) & \dots & p(1, 1) & \\
 \vdots & \ddots & \vdots & \\
 p(|p|_{row}, |p|_{col}) & \dots & p(1, |p|_{col}) & \\
 \end{array}$$

The *column concatenation* \oplus is defined, for all pictures p, q such that $|p|_{row} = |q|_{row}$, written $p \oplus q$ or also written $p q$, as:

$$\begin{array}{ccccccc}
& p(1, 1) & \dots & p(1, |p|_{col}) & q(1, 1) & \dots & q(1, |q|_{col}) \\
p \oplus q = & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
& p(|p|_{row}, 1) & \dots & p(|p|_{row}, |p|_{col}) & q(|q|_{row}, 1) & \dots & q(|q|_{row}, |q|_{col})
\end{array}$$

The row concatenation \oplus for pictures p, q , written $p \oplus q$ or $\overset{p}{q}$, is defined analogously. The empty picture λ is the neutral element for both concatenation operations.

Rotations and the two kinds of concatenations can be extended to picture languages as usual.

Given a picture language L , the column concatenation closure of L , written $L^{*\oplus}$ is the closure of the set L under the column concatenation operation, i.e., $\bigcup_{i \geq 0} L^{i\oplus}$, where $L^{0\oplus} = \lambda$, $L^{i\oplus} = L \oplus (L^{(i-1)\oplus})$ for $i > 0$. The row concatenation closure of L , written $L^{*\ominus}$, is the closure of the set L under the row concatenation operation.

In this paper we define $L^{*,*}$ as in Simplot [10]. To describe this operator we first need to introduce the concepts of subpicture and of partition, which will be also important in the next sections.

Definition 1. Let p, q be pictures. For every i, j , with $1 \leq i \leq |p|_{row}$, $1 \leq j \leq |p|_{col}$, q is a *subpicture* of p at position (i, j) , written $q \trianglelefteq_{(i,j)} p$, if $1 \leq |q|_{row} \leq |p|_{row} - i + 1$, $1 \leq |q|_{col} \leq |p|_{col} - j + 1$, and $q(x, y) = p(i + x - 1, j + y - 1)$ for all $1 \leq x \leq |q|_{row}$, $1 \leq y \leq |q|_{col}$. If there are i, j such that $(q \trianglelefteq_{(i,j)} p)$ then we also write $q \trianglelefteq p$ and define $coord_{(i,j)}(q, p) = \{(x, y) : i \leq x < i + |q|_{row}, j \leq y < j + |q|_{col}\}$. Conventionally, $coord_{(i,j)}(q, p) = \emptyset$, if q is not a subpicture of p at position (i, j) . If q coincides with p we write $coord(p)$ instead of $coord_{(1,1)}(p, p)$.

Definition 2. Let $\Pi(p) = \{(p_1, i_1, j_1), \dots, (p_n, i_n, j_n)\}$, with $n \geq 1$, where for each m , $1 \leq m \leq n$, p_m is a picture such that $p_m \trianglelefteq_{(i_m, j_m)} p$. The set $\Pi(p)$ is a *partition* of p in subpictures p_1, \dots, p_n if the set $\{coord_{(i_m, j_m)}(p_m, p) : (p_m, i_m, j_m) \in \Pi(p)\}$ is a partition of $coord(p)$. Partition $\Pi(p)$ is called *homogeneous* if each p_m is homogeneous. Given $L \subseteq \Sigma^{+,+}$, a set $\Pi = \{(p, \Pi(p)) : p \in L\}$, where each $\Pi(p)$ is a (homogeneous) partition of $p \in L$, is called a (homogeneous) *partition set* of L .

We omit the detailed definition of Simplot's operator. Informally, $p \in L^{+,+}$ iff there exists a partition of p where each subpicture is in L . Let $L^{*,*}$ be the set $L^{+,+} \cup \{\lambda\}$. For example:

$$\begin{array}{c}
a \ a \ b \\
b \ e \ b \\
b \ b \ c
\end{array}
\in \left\{ \begin{array}{ccc}
& b & d \\
a \ a, & & b \ c, \\
& b & d \\
& & e
\end{array} \right\}^{*,*}$$

If all the pictures of L have the same size, then $(L^{*\oplus})^{*\ominus} = (L^{*\ominus})^{*\oplus} = L^{*,*}$.²

Another useful operator introduced by Simplot is the pixel-wise Cartesian product \otimes . For two pictures p, q of the same size, $p \otimes q$ is the picture such that $|p \otimes q| = |p| = |q|$ and

$$\forall 1 \leq i \leq |p|_{row}, \forall 1 \leq j \leq |p|_{col} : (p \otimes q)(i, j) = (p(i, j), q(i, j)).$$

Clearly, if $p \in \Sigma^{+,+}, q \in \Delta^{+,+}$ then $p \otimes q \in (\Sigma \times \Delta)^{+,+}$. This operator is naturally extended to picture languages: let $L \subseteq \Sigma^{+,+}, L' \subseteq \Delta^{+,+}$, then

$$L \otimes L' = \{q : \exists p \in L, \exists p' \in L', q = p \otimes p'\}.$$

We list here the essential definitions of local languages and tiling systems. The *bordered version* of picture p is the picture \bar{p} of size $(|p|_{row} + 2, |p|_{col} + 2)$ obtained by bordering p with a special *boundary symbol* $\# \notin \Sigma$:

$$\bar{p} = \begin{array}{cccccc} \# & \# & \dots & \# & \# \\ \# & p(1, 1) & \dots & p(1, |p|_{col}) & \# \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \# & p(|p|_{row}, 1) & \dots & p(|p|_{row}, |p|_{col}) & \# \\ \# & \# & \dots & \# & \# \end{array}$$

For $p \in \Sigma^{+,+}$ the *set of subpictures* of size (h, k) of p is:

$$B_{h,k}(p) = \{q \in \Sigma^{h,k} : q \sqsubseteq p\}.$$

For a picture language L , $B_{h,k}(L) = \{B_{h,k}(p) : p \in L\}$. If $\theta \subseteq (\Sigma \cup \#)^{(2,2)}$ then the elements of θ are called *tiles* and $LOC(\theta)$ is the set of pictures $p \in \Sigma^{+,+}$ such that $B_{2,2}(\bar{p}) \subseteq \theta$. Sets of tiles are denoted in the following with the symbols θ, θ', \dots . A language $L \subseteq \Sigma^{+,+}$ is *local* if $L = LOC(\theta)$ for some $\theta \subseteq (\Sigma \cup \#)^{(2,2)}$.

The family of local languages over any alphabet will be called LOC for short.

Definition 3. [4] A *tiling system* (TS) is the 4-ple $T = (\Sigma, \Gamma, \theta, \pi)$, where:

Σ and Γ are two finite alphabets,

$\pi : \Gamma \rightarrow \Sigma$ is a mapping,

θ is a finite set of 2×2 tiles over the alphabet $\Gamma \cup \{\#\}$.

The language $L(T) = \pi(LOC(\theta))$ is the *language defined by the TS* T .

² The definition of [10] differs from the one of [4], where $L^{*,*}$ instead denotes the concatenation closure of language L .

The languages over finite alphabets defined by tiling systems constitute the family *TS-REC* of *TS-recognizable languages* (shortly, TS languages) on Σ . TS-REC is closed under: intersection, union, projection, horizontal and vertical concatenation [4], Simplot's operator $^{+,+}$, and Cartesian product \otimes [10].

3 Substitutions

A well-known and widely useful concept in 1D languages is substitution, which assign languages to letters of an alphabet. The mapping is naturally extended to strings and languages too. For example, if a substitution σ maps a into 01^* , and b into 001^+ , then $\sigma(ab)$ is the language 01^*001^+ . For picture languages, it is straightforward to similarly define a substitution as a mapping from pixels to 2D languages:

Definition 4. Given two finite alphabets Σ and Δ , a *substitution* from Δ to Σ is a mapping $\sigma : \Delta \rightarrow 2^{\Sigma^{+,+}}$. Moreover, σ is a *TS substitution* if $\sigma(a)$ is a TS language for every $a \in \Delta$.

But a difficulty hinders the extension of the mapping to pictures, because of the so-called shearing problem of picture languages: a pixel in a picture cannot be replaced by a larger picture without disrupting the array structure. The next definitions overcome the problem by replacing an a -homogeneous subpicture p_a , at position (i, j) , of p with a picture $q \in \sigma(a)$ of identical size, i.e., with $|q| = |p_a|$. This definition, however, is not equivalent to the traditional notion of substitution when applied to strings.

Definition 5. If p, q, q' are pictures and (i, j) is a position in p , with $q \trianglelefteq_{(i,j)} p$, and $|q| = |q'|$, then $p[q'/q]_{(i,j)}$ denotes the picture obtained by replacing the occurrence of q at position (i, j) in p with q' , i.e., $p[q'/q]_{(i,j)}(i+x-1, j+y-1) = q'(x, y)$ for all $1 \leq x \leq |q|_{row}, 1 \leq y \leq |q|_{col}$.

Definition 6. Let $\sigma : \Delta \rightarrow 2^{\Sigma^{+,+}}$ be a substitution. Given a picture $p \in \Delta^{+,+}$, let $\Pi(p) = \{(p_1, i_1, j_1), \dots, (p_n, i_n, j_n)\}$, with $n \geq 1$, be a homogeneous partition of p , where each p_m , $1 \leq m \leq n$, is a d_m -picture for some $d_m \in \Delta$. Then the *substitution of $p \in \Delta^{+,+}$ induced by $\Pi(p)$* is the language $\sigma_{\Pi(p)}(p) = \{p[r_1/p_1]_{(i_1, j_1)} \dots [r_n/p_n]_{(i_n, j_n)} : r_m \in \sigma(d_m), 1 \leq m \leq n\}$. If $L \subseteq \Delta^{+,+}$ and Π is a homogeneous partition set of L , then the *substitution of L induced by the homogeneous partition set Π* is the language $\sigma_{\Pi}(L) = \{\sigma_{\Pi(p)}(p) : p \in L\}$.

In general, there are many homogeneous partitions of a picture, and accordingly many different ways to apply a substitution. In this paper, we consider three different cases, called block substitution, universal substitution and corner substitution.

3.1 Block languages and block substitutions

The next technical steps allow us to mark with distinct symbols 1,2,3,4 the four corners of a subpicture (leaving the inner pixels unmarked or more precisely marked by a dot) in order to use it as a partition block.

Let \mathcal{M} be the set $\{\cdot, \overset{1}{\cdot}, \underset{4}{\cdot}, \overset{2}{\cdot}, \underset{3}{\cdot}, \underset{43}{\cdot}, \overset{12}{\cdot}, \overset{1}{\underset{4}{\cdot}}, \overset{2}{\underset{3}{\cdot}}, \overset{12}{\underset{43}{\cdot}}\}$. The block version of a finite alphabet Δ is the set $\Delta \times \mathcal{M}$. When drawing pictures, given $x \in \Delta$, we will write $x, \overset{1}{x}, \underset{4}{x}, x^2, x_3, \underset{4}{x}_3, \overset{1}{x}^2, \overset{1}{\underset{4}{x}}, x_3^2, \overset{1}{\underset{4}{x}}_3$ instead of

$$(x, \cdot), (x, \overset{1}{\cdot}), (x, \underset{4}{\cdot}), (x, \overset{2}{\cdot}), (x, \underset{3}{\cdot}), (x, \underset{43}{\cdot}), (x, \overset{12}{\cdot}), (x, \overset{1}{\underset{4}{\cdot}}), (x, \overset{2}{\underset{3}{\cdot}}), (x, \overset{12}{\underset{43}{\cdot}}).$$

Definition 7. Consider a d -homogeneous picture p , for some $d \in \Delta$. The *blocking* of p , written $\square(p)$, is a picture $q \in (\Delta \times \mathcal{M})^{+,+}$ defined as:

- $q = (d, \overset{12}{\underset{43}{\cdot}})$, if $|p| = (1, 1)$;
- $q = (d, \overset{12}{\cdot}) \ominus (d, \cdot)^{(|p|_{row}-2)^\ominus} \ominus (d, \underset{43}{\cdot})$, if $|p|_{col} = 1, |p|_{row} > 1$;
- $q = (d, \overset{1}{\underset{4}{\cdot}}) \oplus (d, \cdot)^{(|p|_{col})^\oplus} \oplus (d, \overset{2}{\underset{3}{\cdot}})$, if $|p|_{row} = 1, |p|_{col} > 1$;
- $q = \left((d, \overset{1}{\cdot}) \oplus (d, \cdot)^{(|p|_{col}-2)^\oplus} \oplus (d, \overset{2}{\cdot}) \right) \ominus$
 $\left((d, \cdot)^{|p|_{row}-2, |p|_{col}} \right) \ominus$
 $\left((d, \underset{4}{\cdot}) \oplus (d, \cdot)^{(|p|_{col}-2)^\oplus} \oplus (d, \underset{3}{\cdot}) \right)$
 otherwise.

For example, if p is the picture $b \ b \ b$ then $\square(p)$ is $\overset{1}{\underset{4}{b}} \ b \ b \ b_3^2$.

$$\text{If } p \text{ is } \begin{array}{c} b \ b \ b \ b \\ b \ b \ b \ b \end{array} \text{ then } \square(p) \text{ is } \begin{array}{c} \overset{1}{b} \ b \ b \ b^2 \\ \underset{4}{b} \ b \ b \ b_3 \end{array}.$$

The blocking $\square(L)$ of a language L of homogeneous pictures is $\{\square(p) : p \in L\}$. For every $d \in \Delta$, a *block d -picture* is a picture $p \in \square(d^{+,+})$; p is also called *block homogeneous*. The *universal block language* over an alphabet Δ , denoted as $\Delta^{\square+,+}$ is the set $(\bigcup_{a \in \Delta} \square(a^{+,+}))^{+,+}$.

A block picture is an element of $\Delta^{\square+,+}$, and a block language is a subset of $\Delta^{\square+,+}$.

Clearly, $\square(d^{+,+})$ is a local language, therefore from closure of TS-REC under union and Simplot's $^{+,+}$ operator, the universal block language is in TS-REC.

Given two finite alphabets Σ', Σ'' , let $|\Sigma' : \Sigma' \times \Sigma'' \rightarrow \Sigma', |\Sigma'' : \Sigma' \times \Sigma'' \rightarrow \Sigma''$ be such that for all $a \in \Sigma', b \in \Sigma'', |\Sigma'(a, b) = a, |\Sigma''(a, b) = b$. These component projections may be extended as usual from pixels to pictures and to languages.

By closure under projection, it follows that if a block language L is in TS-REC, also $|\Delta(L)$ is so.

Definition 8. Let $p \in (\Delta \times \mathcal{M})^{+,+}$ be a block picture on the alphabet Δ , and let $\Pi(p) = \{(p_1, i_1, j_1), \dots, (p_n, i_n, j_n)\}$, $n \geq 1$, be a partition of p . If for every m ,

$1 \leq m \leq n$, p_m is a block homogeneous picture then $\Pi_N(p) = \{(|_\Delta(q), i, j) : (q, i, j) \in \Pi(p)\}$ is called the *natural partition* of $|_\Delta(p)$.

Notice that for every block picture p on the alphabet Δ there is one, and only one, natural partition $\Pi_N(p)$ of $|_\Delta(p)$, since the partition of p in block homogeneous subpictures is unique.

Example 1. Let $\Delta = \{\underline{black}, \underline{white}\}$, $\Sigma = \{\underline{niger}, \underline{albus}\}$, and consider the local language $L_1 \subset \Sigma^{+,+}$ of checkered grids, exemplified by the pictures:

$$p_1 = \begin{array}{cccc} b & w & w & b & b \\ b & w & w & b & b \\ w & b & b & w & w \end{array} \quad p_2 = \begin{array}{cccc} & & & & w & w & b & b \\ & & & & w & w & b & b \end{array}$$

The next pictures are examples of natural partitions of p_1 and p_2 :

$$q_1 = \begin{array}{cccc} {}^1b^2 & {}^1w & w^2 & {}^1b & b^2 \\ {}_4b_3 & {}_4w & w_3 & {}_4b & b_3 \end{array} \quad r_1 = \begin{array}{cccc} {}^1b_3^2 & {}^1w & w^2 & {}^1b & b^2 \\ {}_4b_3^2 & {}_4w & w_3 & {}_4b & b_3 \end{array} \quad q_2 = \begin{array}{cccc} {}^1w & w^2 & {}^1b & b^2 \\ {}_4w & w_3 & {}_4b & b_3 \end{array}$$

Definition 9. Given p and $\Pi_N(p)$ as in Def. 8 and a substitution $\sigma : \Delta \rightarrow 2^{\Sigma^{+,+}}$, the *block substitution* of p , denoted $\sigma_B(p)$, is $\sigma_{\Pi_N(p)}(|_\Delta(p))$. Conventionally, if p is not a block picture then define $\sigma_B(p) = p$. If L is a block language, the *block substitution* of L , written $\sigma_B(L)$, is $\{q : \exists p \in L, q \in \sigma_B(p)\}$. Conventionally, if L is not a block language then $\sigma_B(L) = L$.

Notice that if $\sigma(a)$ is a block language for every $a \in \Delta$, i.e., $\sigma(a) \in \Gamma^{\square^{+,+}}$ for some alphabet Γ , then if L is a block language, also $\sigma_B(L)$ is a block language. If $\sigma(a)$ is a TS-REC language for every $a \in \Delta$ then $\sigma_B(L)$ is called a TS block substitution.

Example 2. Consider the pictures of Ex. 1 and the substitution $\sigma(b) = \{n^{m,m} : m \geq 1\}$, $\sigma(w) = \{a^{m,m} : m \geq 1\}$ Applying the block substitution we obtain:

$$\sigma_B(q_1) = \emptyset, \quad \sigma_B(r_1) = \begin{array}{cccc} n & a & a & n & n \\ n & a & a & n & n \end{array} = s_1, \quad \sigma_B(q_2) = \begin{array}{cccc} a & a & n & n \\ a & a & n & n \end{array} = s_2$$

Lemma 1. *The family of TS languages is closed under TS block substitution.*

PROOF. For the proof it remains to consider only TS languages that are block languages, since otherwise a block substitution is as an identity transformation. Assume that L is a block language on the alphabet Δ . Therefore, $L \subseteq (\Delta \times \mathcal{M})^{+,+}$. Let $\sigma : \Delta \rightarrow 2^{\Sigma^{+,+}}$ be a TS substitution, where Σ is an alphabet. Let $|_{\Delta \times \mathcal{M}}$ and $|_\Sigma$ denote the component projections from $\Delta \times \mathcal{M} \times \Sigma$ to, resp., $\Delta \times \mathcal{M}$ and Σ , and for every $d \in \Delta, q \in \Sigma^{+,+}$ let $\rho_d(q) = \square(d^{|q|}) \otimes q$.

For example, if $\Sigma = \{a, b\}$, $q = \begin{array}{ccc} a & a & a \\ b & a & a \end{array}$, $d \in \Delta$ then:

$$\rho_d(q) = \begin{array}{ccc} ({}^1d, a) & (d, a) & (d^2, a) \\ ({}_3d, b) & (d, a) & (d_4, a) \end{array}, \quad |_{\Delta \times \mathcal{M}}(\rho_d(q)) = \begin{array}{ccc} {}^1d & d & d^2 \\ {}_3d & d & d_4 \end{array}.$$

For a picture language L' on Σ let $\rho_d(L') = \{\rho_d(p) : p \in L'\}$

We claim that

$$\sigma_B(L) = |_{\Sigma} \left((L \otimes \Sigma^{+,+}) \cap \left(\bigcup_{d \in \Delta} \rho_d(\sigma(d)) \right)^{+,+} \right)$$

The thesis follows then immediately by closure of TS languages under projection by mapping, Simplot's $^{+,+}$ and \otimes operators, union, and intersection. In the remainder of the proof, for simplicity we use $\sigma(p), \sigma(L), \dots$ to stand for $\sigma_B(p), \sigma_B(L), \dots$. We first prove that: $\sigma(L) \subseteq |_{\Sigma} \left((L \otimes \Sigma^{+,+}) \cap \left(\bigcup_{d \in \Delta} \rho_d(\sigma(d)) \right)^{+,+} \right)$. Let $p \in \sigma(L)$. Hence, there exists $q \in L$ such that $p \in \sigma(q)$. There is $n > 0$ such that $\{(q_i, m_i, n_i) : 1 \leq i \leq n\}$ is the natural partition of q . Therefore, each $q_i = \square \left(d_i^{h_i, k_i} \right)$ for some $d_i \in \Delta, h_i > 0, k_i > 0$. By definition of block substitution, p is obtained from q by replacing each q_i in q with a $p_i \in \sigma(q_i)$ such that $|p_i| = |q_i|$. But $p_i \in |_{\Sigma}(q_i \otimes \Sigma^{+,+})$, since the latter is the set of pictures in $\Sigma^{+,+}$ with the same size of q_i and hence of p_i . Also, there is r on the alphabet $\Delta \times \mathcal{M} \times \Sigma$ such that $r \in (q \otimes \Sigma^{+,+})$ and $p = |_{\Sigma}(r)$. By q being decomposed as described above, and since $q = |_{\Delta \times \mathcal{M}}(r)$, $r \in \left(\bigcup_{1 \leq i \leq n} \rho_{d_i}(\sigma(d_i)) \right)^{+,+} \subseteq \left(\bigcup_{d \in \Delta} \rho_d(\sigma(d)) \right)^{+,+}$. Moreover, $r \in (q \otimes \Sigma^{+,+}) \subseteq (L \otimes \Sigma^{+,+})$. Hence, $r \in (L \otimes \Sigma^{+,+}) \cap \left(\bigcup_{d \in \Delta} \rho_d(\sigma(d)) \right)^{+,+}$. The thesis then follows since $p = |_{\Sigma}(r)$.

We now prove that: $|_{\Sigma} \left((L \otimes \Sigma^{+,+}) \cap \left(\bigcup_{d \in \Delta} \rho_d(\sigma(d)) \right)^{+,+} \right) \subseteq \sigma(L)$.

Let $p \in |_{\Sigma} \left((L \otimes \Sigma^{+,+}) \cap \left(\bigcup_{d \in \Delta} \rho_d(\sigma(d)) \right)^{+,+} \right)$. Therefore, there exist $q \in L, r \in (L \otimes \Sigma^{+,+})$ such that $p = |_{\Sigma}(r), q = |_{\Delta \times \mathcal{M}}(r), r \in \left(\bigcup_{d \in \Delta} \rho_d(\sigma(d)) \right)^{+,+}$. Since $q = |_{\Delta \times \mathcal{M}}(r)$, there exist $n \geq 1$ homogeneous subpictures of r , denoted by r_1, \dots, r_n , and subpictures of q , denoted by q_1, \dots, q_n , such that: for every $i, 1 \leq i \leq n$, $r_i \in (q_i \otimes \Sigma^{+,+})$ and there exist $h_i, k_i \geq 1$ and $d_i \in \Delta \times \mathcal{M}$ with $q_i = \square \left(d_i^{h_i, k_i} \right)$. By definition of $|_{\Delta \times \mathcal{M}}$, and since r is also in $\left(\bigcup_{d \in \Delta} \rho_d(\sigma(d)) \right)^{+,+}$, then $r_i \in \rho_{d_i}(\sigma(d_i))$. Therefore, $|_{\Sigma}(r_i) \in \sigma(d_i)$. Hence, $p = |_{\Sigma}(r)$ may be obtained by replacing each subpicture of the form $\square \left(d_i^{+,+} \right)$ in q with $|_{\Sigma}(r_i) \in \sigma(d_i)$. Therefore, $p \in \sigma(q) \subseteq \sigma(L)$. \square

By suitably blocking the homogeneous subpictures defined by a homogeneous partition, it follows:

Corollary 1. *TS-REC is closed under a TS substitution induced by a homogeneous partition set.*

3.2 Universal substitutions

Next, instead of imposing a given partition on a picture, we extend the substitution operation in order to consider all possible partitions in homogeneous subpictures.

Definition 10. Given a substitution $\sigma : \Delta \rightarrow 2^{\Sigma^{+,+}}$, the *universal substitution* of $p \in \Delta^{+,+}$ is $\sigma_U(p) = \bigcup_{\Pi(p)} \sigma_{\Pi(p)}(p)$ a homogeneous partition of p .

For a language L , $\sigma_U(L) = \{q : q \in \sigma_U(p) \text{ for some } p \in L\}$ is the universal substitution of L .

Example 3. Consider the previous examples 1 and 2. For every picture p , let $\Pi(p)$ be the homogeneous partition of p into homogeneous blocks of unitary side. Then $\sigma_{\Pi}(L_1) = h(L_1)$, where $h(b) = n, h(w) = a$ is a mapping. Any other homogeneous partition $\Pi'(p)$ is such that either $\sigma_{\Pi'(p)}(p) = h(p)$ or $\sigma_{\Pi'(p)}(p)$ is undefined. Hence, $\sigma_U(L_1) = h(L_1)$.

Theorem 1. *TS-REC is closed under universal substitution.*

PROOF. Let L be a TS-recognizable language over the alphabet Σ . From [10], we know that also $(L \otimes \mathcal{M}^{+,+})$ is in TS-REC. Moreover, the block language $\Sigma^{\square+,+}$ is TS-recognizable. Hence, also $L' = (L \otimes \mathcal{M}^{+,+}) \cap \Sigma^{\square+,+}$ is TS-recognizable, by closure of TS-REC under intersection. Let $\sigma : \Sigma \rightarrow 2^{\Delta^{+,+}}$ be a substitution. Thanks to Lemma 1, $\sigma_B(L')$ is in TS-REC. By definition of block substitution, for each block picture $p' \in L'$, whose partition in block homogeneous subpictures is $\{(p_m, i_m, j_m) : 1 \leq m \leq n\}$, $\sigma_B(p') = \sigma_{\Pi(p')}(|_{\Sigma}(p')) \subseteq \sigma_U(|_{\Sigma}(p'))$, with $\Pi(p') = \{(|_{\Sigma}(p'_m, i_m, j_m) : 1 \leq m \leq n\}$. Hence, $\sigma_B(L') \subseteq \sigma_U(L)$. Conversely, for each $p \in L$ and for each partition $\Pi(p) = \{(p_m, i_m, j_m) : 1 \leq m \leq n\}$ in homogeneous subpictures p_1, \dots, p_m , there is, by construction of L' , a picture p' whose partition in block homogeneous subpictures is $\{(p'_m, i_m, j_m) : |_{\Sigma}(p'_m) = p_m, 1 \leq m \leq n\}$. Hence $\sigma_{\Pi(p)}(p) = \sigma_B(p')$ and $\sigma_U(L) = \sigma_B(L')$. \square

3.3 Nested iterated substitutions

Nested iterated substitutions for string languages were introduced by Greibach in [5]. They are more powerful than substitutions: for instance, nested iterated substitutions of regular sets define context-free languages. We now want to define a version of nested iterated substitutions for picture languages, which preserves the TS-recognizability of a language. This is very useful in connection with TRG, as

shown in Section 4. In particular, we only allow iterated substitutions into a subpicture that includes one (or more) of the four corners of the picture. This is essential to extend the traditional result that non-self-embedding context-free grammars only define regular languages [3].

For a substitution $\sigma : \Delta \rightarrow 2^{\Delta^{+,+}}$ and a picture language $L \subseteq \Delta^{+,+}$, define $\sigma^0(L) = L$, $\sigma^i(L) = \sigma(\sigma^{i-1}(L))$ for every $i > 0$.

Definition 11. A substitution $\sigma : \Delta \rightarrow 2^{\Delta^{+,+}}$ is *nested* if $a^{+,+} \subseteq \sigma(a)$ for every $a \in \Delta$. Given a homogeneous partition set Π defined for every picture $p \in \Delta^{+,+}$, $\sigma_{\Pi}^*(L) = \bigcup_{i \geq 0} \sigma_{\Pi}^i(L)$ is called a *nested iterated substitution* of a language L .

If a substitution is nested, then for every language L and for every $i \geq 0$ it is $\sigma_{\Pi}^i(L) \subseteq \sigma_{\Pi}^{i+1}(L)$. In particular it is $L \subseteq \sigma_{\Pi}(L)$.

Next we focus on the subpictures placed in a corner, and on the case when they may only contain characters from a subalphabet. A *corner* of a picture p is an element of the set $\{1, |p|_{row}\} \times \{1, |p|_{col}\}$.

Given two disjoint alphabets Ξ, Δ , we introduce the following notation: $(\Xi, \Delta)_1 = \bigcup_{X \in \Xi} (X^{*,*} \ominus \Delta^{*,*}) \oplus \Delta^{*,*}$, and for $1 < i \leq 4$, $(\Xi, \Delta)_i = rot^{i-1}((\Xi, \Delta)_1)$.

A subset of $(\Xi, \Delta)_i$ is called a $(\Xi, \Delta)_i$ -*corner language*.

Notice that a picture in $(\Xi, \Delta)_1$ is made of a X -homogeneous top-left rectangle, for some $X \in \Xi$, while the remaining pixels are in Δ ; both parts may be empty.

Definition 12. If $p \in (\Xi, \Delta)_i$, then the *corner partitioning* $\Pi_C(p)$ of p is the partition induced by considering as subpictures all individual pixels of p in Δ , and the remaining X -homogeneous subpicture, where $X \in \Xi$.

An illustration of corner partitioning with $X \in \Xi$, $a, b \in \Delta$, in which every partition is marked with a box, is the following:

X	X	a	a
X	X	a	a
b	b	b	a

Definition 13. A $(\Xi, \Delta)_i$ -*corner substitution*, where $1 \leq i \leq 4$, is a substitution $\sigma : \Delta \cup \Xi \rightarrow 2^{(\Delta \cup \Xi)^{+,+}}$, such that for every $a \in \Delta$, $\sigma(a) = a^{+,+}$, and for every $X \in \Xi$, $\sigma(X)$ is a $(\Xi, \Delta)_i$ -corner language. For every corner picture $p \in (\Xi, \Delta)_i$, the corner substitution of p is $\sigma_C(p) = \sigma_{\Pi_C(p)}(p)$, where $\Pi_C(p)$ is the corner partitioning of p . For a language $L \subseteq (\Xi, \Delta)_i$, $\sigma_C(L) = \{q : \exists p \in L, q \in \sigma_C(p)\}$.

A $(\Xi, \Delta)_i$ -corner substitution is defined only on $(\Xi, \Delta)_i$ -corner languages. Its result is again a $(\Xi, \Delta)_i$ -corner language. A corner substitution σ as in the above definition is a TS corner substitution if for every $X \in \Xi$, $\sigma(X)$ is a TS language. By Corollary 1, TS-REC is closed also under TS corner substitution.

A corner picture can be described as a row concatenation followed by a column

concatenation, e.g., a corner picture $p \in \sigma_C(X^{*,*})$ is of the form $(q \oplus p_r) \oplus p_c$, with $q \in X^{*,*}$ and $p_r, p_c \in \Delta^{*,*}$. A nested iterated corner substitution is then akin to the closure under this double concatenation. This is strictly related to a very similar concatenation operation, studied by Matz [7], in so called $\cap - REG^{ROP}$ expressions, where a concatenation with a row of height 1 is followed by a concatenation with a column of width 1. The results in [7] do not directly imply closure of TS languages under nested iterated TS corner substitutions, which is instead proved next.

Our proof of closure is conceptually similar to the following (traditional) proof that TS languages are closed under row concatenation closure. Consider the TS $(\Sigma, \Gamma, \theta, \pi)$, and call L the language it defines. Then, one can define another tiling system $(\Sigma, \Gamma \cup \Gamma', \theta \cup \theta', \pi')$ where: Γ' is a marked copy (often called a *coloring*) of Γ ; π' extends π , so that $\pi'(a') = \pi(a)$ for every $a \in \Gamma$, with a' the marked copy of a ; $\theta' \subseteq (\Gamma' \cup \#)^{2,2}$ such that $\pi'(LOC(\theta)) = \pi'(LOC(\theta')) = \pi'(LOC(\theta \cup \theta')) = L$. Then let

$$\theta_1 = B_{2,2} \left(\overline{LOC(\theta) \cup LOC(\theta) \oplus LOC(\theta') \cup LOC(\theta) \oplus LOC(\theta') \oplus LOC(\theta)} \right)$$

(we remind the reader that $B_{2,2}(\overline{L_1})$ is the set of all 2×2 tiles that may occur in the bordering of pictures in language L_1). Due to the use of two alternating alphabets, $\pi'(LOC(\theta_1))$ is $L^{\oplus+}$.

The case with corner substitutions is quite similar, since again one has to consider a (special) concatenation closure. The details are more complex, since one needs four different local alphabets, in order to be able to alternate two pairs of colors (one pair for row concatenations and one pair for column concatenations). Before proving the main theorem, we need the following lemma, whose proof is immediate, allowing the introduction of these different colorings in the tiling system of a TS corner language.

Lemma 2. *If $C \subseteq (\Delta \cup \Xi)^{*,*}$ is a $(\Xi, \Delta)_1$ -corner language, and C is a TS language, then there exists a tiling system $(\Delta \cup \Xi, \Gamma, \theta, \pi)$ such that:*

- (1) $\pi(LOC(\theta)) = C$;
- (2) $\Gamma = \Gamma_{\Xi} \cup \Gamma_{\Delta} \cup \Gamma'_{\Delta} \cup \Gamma''_{\Delta} \cup \Gamma'''_{\Delta}$, where $\Gamma_{\Xi}, \Gamma_{\Delta}, \Gamma'_{\Delta}, \Gamma''_{\Delta}, \Gamma'''_{\Delta}$, are five pairwise disjoint alphabets with $\Gamma'_{\Delta}, \Gamma''_{\Delta}, \Gamma'''_{\Delta}$, made of distinguished marked copies of the symbols in Γ_{Δ} ;
- (3) $\pi(\Gamma_{\Xi}) = \Xi, \pi(\Gamma_{\Delta}) = \pi(\Gamma'_{\Delta}) = \pi(\Gamma''_{\Delta}) = \pi(\Gamma'''_{\Delta}) = \Delta$;
- (4) $LOC(\theta) \subseteq \left(\Gamma_{\Xi}^{*,*} \oplus (\Gamma_{\Delta}^{*,*} \cup \Gamma'_{\Delta}^{*,*}) \right) \oplus (\Gamma''_{\Delta}^{*,*} \cup \Gamma'''_{\Delta}^{*,*})$.

A few more definitions are also needed. The symbols in the marked alphabets of Lemma 2 are denoted, for all $a \in \Gamma_{\Delta}$, with a', a'', a''' respectively. These marked copies of Γ_{Δ} are used for “coloring” the tiles of θ . Let $h : \Gamma - \Gamma_{\Xi} \rightarrow \Gamma - \Gamma_{\Xi}$ be the mapping defined by $h(a) = a', h(a') = a, h(a'') = a''', h(a''') = a''$ for every $a \in \Gamma_{\Delta}$. The mapping h is used to “cycle” between the colors of Γ_{Δ} and those of

Γ'_Δ , which are used to color the rows below a corner, and also to cycle between the colors of Γ''_Δ and those of Γ'''_Δ , which are used to color the columns at the right of a corner.

We also define a column concatenation operation, \ominus_h and a row concatenation operation \oplus_h as follows, to allow the correct alternation of colors: for every picture $p \in \Gamma^{+,+}$, $q \in \Gamma_{\Delta}^{+,+}$, if $\forall j, 1 \leq j \leq |p|_{col}, p(|p|_{row}, j) \notin \Gamma'$ then $p \ominus_h q = p \ominus h(q)$, else $p \ominus_h q = p \ominus q$. For every picture $p \in \Gamma^{+,+}$, $q \in \Gamma_{\Delta}''^{+,+}$, if $\forall i, 1 \leq i \leq |p|_{row}, p(i, |p|_{col}) \notin \Gamma''$ then $p \oplus_h q = p \oplus q$, else $p \oplus_h q = p \oplus h(q)$. This means that if p has, on the lowest row, no symbol in Γ' then \ominus_h concatenates p and $h(q)$, else \ominus_h concatenates p with q . Symmetrically, if p has, on the rightmost column, no symbol in Γ'' then \oplus_h concatenates p and q (i.e., q is colored in Γ''_Δ), else it concatenates p and $h(q)$ (i.e., q is colored in Γ'''_Δ).

Theorem 2. *The family TS-REC is closed under nested iterated TS corner substitution.*

PROOF. Let $\sigma : \Delta \cup \Xi \rightarrow 2^{(\Delta \cup \Xi)^{+,+}}$ be a nested $(\Xi, \Delta)_i$ -corner substitution (for some $i, 1 \leq i \leq 4$) such that $\sigma(X)$ is a TS $(\Xi, \Delta)_i$ -corner language for every $X \in \Xi$.

It remains to show that $\sigma_C^*(\bigcup_{X \in \Xi} X^{+,+})$ is a TS language. The thesis then follows immediately, since given any corner TS language L' , $\sigma_C^*(L')$ is a TS language: let $\tau : \Delta \cup \Xi \rightarrow 2^{(\Delta \cup \Xi)^{+,+}}$ be the corner substitution defined by $\tau(X) = \sigma_C^*(X^{+,+})$ for every $X \in \Xi$; by closure of TS languages under corner substitutions, $\tau_C(L') = \sigma_C^*(L')$ is a TS language. Notice that if L' is not a corner language, then $\sigma_C^*(L') = L'$.

We consider only the case where, for every $X \in \Xi$, $\sigma(X)$ is in $(\Xi, \Delta)_1$. By invariance under rotation, the result also applies to every $(\Xi_i, \Delta)_i$.

A tiling system for $\sigma_C(\bigcup_{X \in \Xi} X^{+,+})$ has the form $T = (\Delta \cup \Xi, \Gamma, \theta, \pi)$, verifying the conditions of Lemma 2. The proof defines a tiling system that recognizes $\sigma_C^*(\bigcup_{X \in \Xi} X^{+,+})$.

We define a family of languages $L_0 \subseteq L_1 \subseteq \dots$ such that $\pi(L_i) = \sigma_C^i(\bigcup_{X \in \Xi} X^{+,+})$. Define $L_0 = \bigcup_{X \in \Xi} X^{+,+}$, and for every $i > 0$:

$$L_i = L_{i-1} \cup \bigcup_{p \in L_{i-1}, (p_r, p_c) \in RC} (p \ominus_h p_r) \oplus_h p_c,$$

with $RC = \{(p_r, p_c) : p_r \in \Gamma_{\Delta}^{*,*}, p_c \in \Gamma_{\Delta}''^{*,*}, \exists \gamma \in \Gamma_{\Xi}^{*,*} : (\gamma \ominus p_r) \oplus p_c \in LOC(\theta)\}$.

The set RC is composed of pairs (p_r, p_c) of (possibly empty) pictures that can be used to enlarge a picture γ (such that $\pi(\gamma) \in X^{*,*}$ with $X \in \Xi$) first by a row concatenation with p_r (over Γ_{Δ}) and then by a column concatenation with p_c

(over Γ''_{Δ}), to obtain a picture in $LOC(\theta)$. Then, $\pi(L_i) = \sigma_C^i(\bigcup_{X \in \Xi} X^{+,+})$ for every $i \geq 0$. We claim that there is tiling system $(\Delta \cup \Xi, \Gamma, \theta_1, \pi)$ that recognizes $\pi(\bigcup_{i \geq 1} L_i) = \sigma^*(\bigcup_{X \in \Xi} X^{+,+})$, proving the main thesis. It suffices to consider two iterations for the L_i 's to obtain all the tiles, i.e., let $\theta_1 = B_{2,2}(\overline{L_2})$. Also, it suffices to prove that for every picture $p \in \bigcup_{i \geq 0} L_i$ if, and only if, $B_{2,2}(\overline{p}) \subseteq \theta_1$, since $p \in \bigcup_{i \geq 0} L_i$ if, and only if, $\pi(p) \in \sigma^*(\bigcup_{X \in \Xi} X^{+,+})$. We first show:

$$p \in \bigcup_{i \geq 0} L_i \Rightarrow B_{2,2}(\overline{p}) \subseteq \theta_1$$

Let $p \in L_j$ for some $j \geq 0$. The proof is by induction on j . If $j \leq 2$ then $p \in L_2$ and $B_{2,2}(\overline{L_2}) = \theta_1$. If $j > 2$, then there are $q \in L_{j-2}$, $(v_r, v_c) \in RC$, $(p_r, p_c) \in RC$ such that $p = (((q \ominus_h v_r) \oplus_h v_c) \ominus_h p_r) \oplus_h p_c$, with $(q \ominus_h v_r) \oplus_h v_c \in L_{j-1}$. By induction hypothesis, $B_{2,2}(((q \ominus_h v_r) \oplus_h v_c)) \subseteq \theta_1$. Moreover, there exists $q_0 \in L_0$ (of the same size of q) such that $(q_0 \ominus_h v_r) \oplus_h v_c \in L_1$ (by definition of corner substitution). Then $((q_0 \ominus_h v_r) \oplus_h v_c) \ominus_h p_r \oplus_h p_c \in L_2$. All tiles in $B_{2,2}(\overline{p})$ are in $B_{2,2}(((q \ominus_h v_r) \oplus_h v_c)) \subseteq \theta_1$ or in $B_{2,2}(((q_0 \ominus_h v_r) \oplus_h v_c) \ominus_h p_r \oplus_h p_c) \subseteq \theta_1$. Hence, $B_{2,2}(\overline{p}) \subseteq \theta_1$.

We now show that: $B_{2,2}(\overline{p}) \subseteq \theta_1 \Rightarrow p \in \bigcup_{i \geq 1} L_i$. It is convenient to define another family of languages:

$$H_0 = L_0, H_1 = (\Gamma_{\Xi}^{*,*} \ominus (\Gamma_{\Delta}^{*,*} \cup \Gamma_{\Delta}^{\prime*,*})) \oplus (\Gamma_{\Delta}^{\prime\prime*,*} \cup \Gamma_{\Delta}^{\prime\prime\prime*,*})$$

and for every $i > 0$:

$$H_i = \bigcup_{p \in H_{i-1}, p_r \in \Gamma_{\Delta}^{*,*}, p_c \in \Gamma_{\Delta}^{\prime\prime*,*}} (p \ominus_h p_r) \oplus_h p_c.$$

Clearly, for $i \geq 1$ $H_i \subseteq H_{i+1}$ and $L_i \subseteq H_i$, while $\bigcup_i H_i$ is a local language defined by $B_{2,2}(\overline{H_2})$. Since $\theta_1 \subseteq B_{2,2}(\overline{H_2})$, if $B_{2,2}(\overline{p}) \subseteq \theta_1$ then $p \in H_j$ for some j . Hence, $p = (r \ominus_h p_r) \oplus_h p_c$, with $r \in H_{j-1}$. We claim that if $B_{2,2}(\overline{p}) \subseteq B_{2,2}(\overline{L_2}) = \theta_1$ and $p \in H_j$, for $j \geq 1$, then $p \in L_j$, and from here the thesis follows. The proof is by induction on $j \geq 0$. The base step $j = 0$ is obvious. Another base step, $j = 1$ is also obvious, since $L_1 = LOC(\theta)$: if $B_{2,2}(\overline{p}) \subseteq \theta$ then $p \in L_1$. For $j = 2$, if $p \in H_2$ and $B_{2,2}(\overline{p}) \subseteq B_{2,2}(\overline{L_2})$ then $p \in L_2$, because the usage of two pairs of alternating alphabets does not allow for a mixing of tiles belonging to different alphabets. If $j > 2$, there exist $p_r, v_r \in \Gamma_{\Delta}^{*,*}$, $p_c, v_c \in \Gamma_{\Delta}^{\prime\prime*,*}$ and $q \in H_{j-2}$ (and hence in L_{j-2} by induction hypothesis) such that $p = ((q \ominus_h v_r) \oplus_h v_c) \ominus_h p_r \oplus_h p_c$. We claim that $(p_r, p_c) \in RC$ and therefore $p \in L_j$. Consider a picture $p' \in H_2$ defined by replacing q in p with a picture q_0 (of the same size of q) in L_0 . The (possible) tiles in $\overline{p'}$ but not in \overline{p} are all in θ_1 : $B_{2,2}(\overline{p'}) - B_{2,2}(\overline{p}) \subseteq B_{2,2}(((q_0 \ominus_h v_r) \oplus_h v_c))$, and $(q_0 \ominus_h v_r) \oplus_h v_c \in L_1$, since $(v_r, v_c) \in RC$. Therefore, also $B_{2,2}(\overline{p'}) \subseteq \theta_1$. Hence,

$p' \in L_2$, since $p' \in H_2$. □

4 Tile Rewriting Grammars and Corner Recursion

In this part we prove some interesting relationships between tile rewriting grammars and TS, which extend in 2D two well known properties relating context-free grammars and regular languages, or, which is the same, push-down and finite automata. Part of the difficulty stems from the lack of a characterization of TRG languages by means of automata. To prove the results we had to take a different course, exploiting the previous closure properties of TS with respect to substitutions.

Next we introduce TRG grammars and languages.

Definition 14. A picture $p \in \Delta^{+,+}$ is called y -convex, $y \in \Delta$, if $\forall x \in \Delta, x \neq y$, none of the tiles: $\left\{ \begin{array}{cc} y & y \\ x & y \end{array}, \begin{array}{cc} x & y \\ y & y \end{array}, \begin{array}{cc} y & x \\ y & y \end{array}, \begin{array}{cc} y & y \\ y & x \end{array} \right\}$ is in p . If $\Delta' \subseteq \Delta$, the picture p is Δ' -convex if it is y -convex for each $y \in \Delta'$. A Δ -convex picture is called a convex picture for short. A language is Δ' -convex if every picture of the language is Δ' -convex.

Definition 15. For every $a \in \Delta$, a *maximal a -subpicture* of p is an a -subpicture $q \trianglelefteq_{(i,j)} p$ such that for every a -subpicture $q' \trianglelefteq_{(i',j')} p$, if $coor_{(i,j)}(q, p) \cap coor_{(i',j')}(q', p) \neq \emptyset$ then $coor_{(i',j')}(q', p) \subseteq coor_{(i,j)}(q, p)$. A subpicture q of p is called a *maximal homogeneous subpicture* of p if there exists $a \in \Delta$ such that q is a maximal a -subpicture.

We state the following simple property without proof.

Proposition 1. *For every Δ' -convex picture $p \in \Delta^{+,+}$, $\Delta' \subseteq \Delta$, there exists one, and only one, set of homogeneous maximal subpictures of p , belonging to $\Delta'^{+,+}$.*

In Tile Rewriting Grammars, we also use tiles to define subpictures belonging to a syntax class. For this purpose we need a different definition of local language, called *restricted local*, which does not rely on the presence of boundary symbols. This is necessary because inner subpictures are not bordered by $\#$. A language L is restricted local if there exists a set of tiles θ on the alphabet Γ such that $p \in L$ iff the following conditions hold:

- if $|p|_{row} > 1, |p|_{col} > 1$ then $B_{2,2}(p) = \theta$;
- if $|p|_{row} = 1, |p|_{col} > 1$ then $B_{1,2}(p) = \theta$;
- if $|p|_{row} > 1, |p|_{col} = 1$ then $B_{2,1}(p) = \theta$.

Unlike the definition of local languages, this definition does not consider bordered pictures, but requires that all tiles occur in a picture.

Although restricted local languages are a subfamily of local languages, TS-REC

languages may be equivalently defined [8] as the projection of the union of restricted local languages.

Let $\text{FIN}(\Delta)$ be the family of finite languages in $\Delta^{+,+}$. Let $\text{LCVX}_{\Delta'}(\Delta)$, $\Delta' \subseteq \Delta$, be the family of Δ' -convex restricted local languages over the alphabet Δ .

The next definition of TRG is equivalent to the original one [8].

Definition 16. A *Tile Rewriting Grammar (TRG)* is a tuple (Σ, N, S, R) , where Σ is the *terminal* alphabet, N is the *nonterminal* alphabet, $S \in N$ is the *starting symbol*, $R \subseteq N \times (\text{LCVX}_N(N \cup \Sigma) \cup \text{FIN}(\Sigma))$ is the set of *rules*.

In the sequel (see Example 4) we will denote the elements of N with upper case letters, and the elements of Σ with lower case letters. Intuitively, a rule of the form (A, Ω) , written $A \rightarrow \Omega$, is used to replace, in a derivation from a picture $p \in (N \cup \Sigma)^{+,+}$, a maximal A -subpicture $q \in A^{+,+}$ of p with a picture $\omega \in \Omega$ of the same size of q . The result is a new picture p' . This rewriting is well-defined if p is N -convex, allowing a unique partitioning in maximal homogeneous subpictures. Observe, however, that the N -convexity of p may not be preserved after rewriting q in p with ω , even though ω is a N -convex picture. For instance, if ω is in $B^{+,+}$, $B \in N$, and in p one pixel containing B (B -pixel for short) touches q , then p' may not be B -convex. Even in the case that p' is N -convex, that is all B -pixels might be conjoined together to make a new maximal B -subpicture, a subsequent derivation would simultaneously rewrite the conjoined pixels, thus violating the idea of context-free rewriting. To avoid these interference effects, we introduce a suitable blocking of pictures.

Definition 17. The N -blocking $\square_N : (\Sigma \cup N)^{+,+} \rightarrow (\Sigma \cup (N \times \mathcal{M}))^{+,+}$ of a picture is defined for every N -convex picture $p \in (\Sigma \cup N)^{+,+}$, as follows. The picture $\square_N(p)$ is obtained from p by replacing, for each $A \in N$, every maximal A -subpicture q of p with $\square(q)$.

Notice that, if $p \in A^{+,+}$, $A \in N$, then $\square_N(p) = \square(p)$.

By Proposition 1, if $p \in \text{LCVX}_N(N \cup \Sigma)$, then there exists a unique picture, $\square_N(p)$, whose block homogeneous N -subpictures are maximal.

Definition 18. Let $G = (\Sigma, N, S, R)$ be a TRG. A *one-step derivation* is a binary relation $\Rightarrow_G \subseteq (\Sigma \cup (N \times \mathcal{M}))^{+,+} \times (\Sigma \cup (N \times \mathcal{M}))^{+,+}$ such that $p \Rightarrow_G p'$ if p and p' have the same size and there exist $A \in N$, $(i, j) \in \text{coord}(p)$ and a block A -picture $r \trianglelefteq_{(i,j)} p$ and a rule in R of the form $A \rightarrow \Omega$, a picture $\omega \in \Omega$, with $|r| = |\omega|$, such that: $p' = p[\square_N(\omega)/r]_{(i,j)}$.

We say that p derives in one step p' if $p \Rightarrow_G p'$. Let $\overset{*}{\Rightarrow}_G$ be the reflexive and transitive closure of \Rightarrow_G . We say that p derives p' if $p \overset{*}{\Rightarrow}_G p'$. Notice that $|p| = |p'|$ and that if p is the N -blocking of a picture, also p' is a N -blocking.

See the example after Definition 21.

Definition 19. The *picture language* defined by a grammar G is $L(G) = \{p \in \Sigma^{+,+} : \square(S^{|p|}) \xrightarrow{*}_G p\}$.

With an abuse of notation, if there exists an X -picture $X^{n,m}$ such that $\square(X^{n,m}) \xrightarrow{*}_G p$, we prefer to write $X \xrightarrow{*}_G p$, since the blocking is understood and the size of $X^{n,m}$ is uniquely determined by the size of p . Accordingly, the definition of $L(G)$ may be written as: $\{p \in \Sigma^{+,+} : S \xrightarrow{*}_G p\}$.

It is known that the family of TRG languages strictly includes the TS-REC family. The next developments constrain the form of derivations in order to match the capacity of TS.

The first constraint removes altogether recursive derivations. We recall that non-recursive context-free grammars generate finite languages, but, for a meaningful comparison with TRG, the grammar rules should allow local regular languages in the right parts. Such extension of course does not enlarge the family of generated languages. By constraining the extended context-free grammars to be non-recursive, the family of regular languages is obtained.

A straightforward generalization of the concept of non-recursive context-free grammar is defined next.

Definition 20. Given a TRG G , a nonterminal symbol A is called *non-recursive* if there is no derivation of the form $A \xrightarrow{*}_G p$ where p has a block A -subpicture. If the property holds for all nonterminals, the grammar is called non-recursive.

Two symbols $A, B \in N$ are *mutually recursive* if there exist two derivations $A \xrightarrow{*}_G p, B \xrightarrow{*}_G q$ where p has a block B -subpicture and q has a block A -subpicture.

In order to prove that a non-recursive TRG only generates a TS language, it helps to the concept of derivation by level, to combine independent one-step derivations into a macro step.

Definition 21. A *one-level derivation* is a binary relation $\Rightarrow_G \subseteq (\Sigma \cup (N \times \mathcal{M}))^{+,+} \times (\Sigma \cup (N \times \mathcal{M}))^{+,+}$ such that $p \Rightarrow_G p'$ if p and p' have the same size and, denoting with $r_m \triangleleft_{(i_m, j_m)} p, 1 \leq m \leq n$ the block homogeneous subpictures of p , for each $m, 1 \leq m \leq n$, and for each $A_m \in N$ such that r_m is a block A_m -picture there exist rules in R of the form $A_m \rightarrow \Omega_m$ and pictures $\omega_m \in \Omega_m$ with $|r_m| = |\omega_m|$, such that:

$$p' = p [\square_N(\omega_1)/r_1]_{(i_1, j_1)} [\square_N(\omega_2)/r_2]_{(i_2, j_2)} \dots [\square_N(\omega_n)/r_n]_{(i_n, j_n)}$$

We say that p derives p' in one level if $p \Rightarrow_G p'$. Let $\xrightarrow{*}_G$ be the reflexive and transitive closure of \Rightarrow_G . We say that p derives p' by level if $p \xrightarrow{*}_G p'$. Clearly, if $p \Rightarrow_G p'$ then $p \xrightarrow{*}_G p'$.

The following figure shows a derivation and a one-level derivation from the picture

given in the middle, using the rule $A \rightarrow B^{+,+} \oplus a^{+,+}$:

$$\begin{array}{ccccc} b & b & b & b & \frac{1}{4}B_3^2 a & b & b & b & b & \frac{1}{4}A A_3^2 & b & b & b & b & \frac{1}{4}A A_3^2 \\ {}^1B & B^2 & a & b & b & b & \Leftarrow & {}^1A & A & A^2 & b & b & b & \Rightarrow & {}^1B & B^2 & a & b & b & b \\ {}_4B & B_3 & a & b & b & b & & {}_4A & A & A_3 & b & b & b & & {}_4B & B_3 & a & b & b & b \end{array}$$

Theorem 3. *The family of languages generated by non-recursive TRG grammars coincides with the family of TS-REC languages.*

PROOF. Let $G = (\Sigma, N, S, R)$ be a non-recursive TRG and X a nonterminal symbol. Let $L_{G,X} = \bigcup_{\Omega: X \rightarrow \Omega \in R} \square_N(\Omega)$. We can define a block substitution $\sigma : \Sigma \cup N \rightarrow 2^{\Sigma \cup N^{+,+}}$ as follows: $\sigma(a) = a^{+,+}$ for $a \in \Sigma$, $\sigma(X) = L_{G,X}$ for $X \in N$. (Strictly speaking, σ is a block substitution only if we assume every terminal symbol is of the form $\frac{1}{4}a_3^2$ for some $a \in \Sigma$.) We claim that $L(G) = \sigma^{|N|}(\square(S^{+,+})) \cap \Sigma^{+,+}$.

First, we prove that $L(G)$ is in TS-REC. One level derivation step $q \Rightarrow_G q'$ replaces every block A -subpicture of q with isometric pictures in $\square_N(\Omega)$, when applying a rule $A \rightarrow \Omega$. Hence, if $p \in L(G)$, being G non-recursive, p is obtained by at most $|N|$ steps of a level derivation from S : $p \in \sigma^{|N|}(\square(S^{+,+})) \cap \Sigma^{+,+}$. So $L(G) \subseteq (\sigma^{|N|}(\square(S^{+,+}))) \cap \Sigma^{+,+}$. Now $L(G)$ is TS-recognizable because $\square(S^{+,+})$ and $\Sigma^{+,+}$ are TS-recognizable, and TS-REC is closed under block substitution and intersection.

The opposite direction is obvious from the proof of Theorem 25 of [8] (showing the inclusion of TS languages in the TRG language family), which is based on defining a non-recursive TRG for every TS language. \square

However, as a consequence of the emptiness problem for local languages (see [4], Theorem 9.1), checking whether a TRG is recursive is not decidable. The proof of Theorem 9.1 reduces the problem of emptiness of a local language to the termination of a Turing machine M . The same result holds for restricted local languages: tiles without borders can ensure that the initial state q_0 of M is only in the first line, and that the final state q_f is only in the last line (obviously M must not have cyclic transitions in q_0 , but we can always modify a Turing machine for this). To prove it, consider a TRG G containing the rules: $A \rightarrow \Omega_1 \mid \dots \mid \Omega_k$, where A is the axiom, $\Omega_i, 1 \leq i \leq k$ define the restricted local language which encodes a computation of M (as in Theorem 9.1), and A is used also to represent q_f . Then, checking the recursiveness of G , means checking the termination of M . Therefore, we propose a sufficient condition of non-recursiveness.

Definition 22. Consider two rules $R_1 = A_1 \rightarrow \Omega_1$ and $R_2 = A_2 \rightarrow \Omega_2$ of a TRG

G . The relation \mathcal{D} is defined as $R_1 \mathcal{D} R_2$ iff A_2 is a symbol used in tiles of Ω_1 . The *dependence relation* between rules is the transitive closure of \mathcal{D} .

A TRG G is *syntactically non-recursive* if it does not contain any rule R , such that R depends on itself.

The last development moves in the direction of finding a restriction on the form of TRG rules that in context-free grammars would correspond to the absence of self-embedding derivations. It is known that such grammars precisely correspond to regular languages [3] (for a more recently account see [1]). The idea is to restrict recursion to occur in a corner of the picture as next defined.

Definition 23. A TRG $G = (\Sigma, N, S, R)$ is a *corner grammar* if there exists a partition of N in sets: $N_1, N_2, N_3, N_4, \bar{N}$ such that:

- (1) \bar{N} is the set of non-recursive nonterminals of G ;
- (2) for every $i, 1 \leq i \leq 4$, for each $A \in N_i$ if $A \Rightarrow_G^* p$, then p is a $(N_i, \Sigma \cup N - N_i)_i$ -corner picture;
- (3) for every $i \neq j, 1 \leq i, j \leq 4$, for every $A \in N_i, B \in N_j$, A and B are not mutually recursive.

Clearly, a non-recursive TRG is a special case of corner grammar (with $N_i = \emptyset$ for every $i, 1 \leq i \leq 4$). A corner grammar is a generalization of right-linear or left-linear grammars for the 1D case, and it is the 2D analogous of a 1D grammar where self-embedding is never allowed. We allow, in every non-corner position of a picture, only terminals or those nonterminals that cannot give raise to recursions (i.e., those of \bar{N}), while considering disjoint (possibly empty) nonterminal alphabets for the four corners.

Checking whether a TRG is a corner grammar is not decidable. This can easily be proved along the same lines of the above proof of the undecidability of checking whether a TRG is recursive. It would also be possible to formulate decidable sufficient conditions, which ensure that a TRG is a corner grammar.

Theorem 4. *A corner TRG defines a TS-REC language.*

PROOF. Let $G = (\Sigma, N, S, R)$ be a corner grammar, where $N = N_1 \cup N_2 \cup N_3 \cup N_4 \cup \bar{N}$ is the partition of the definition. As in the proof of Theorem 3, we assume that every $a \in \Sigma$ is actually of the form $\frac{1}{4}a_3^2$. Let $L(G, A) = \{p \in \Sigma^{+,+} : A \xrightarrow{*}_G p\}$. Therefore, $L(G) = L(G, S)$, with either $S \in N_i$ for some $i, 1 \leq i \leq 4$, or $S \in \bar{N}$. We show that $L(G, A)$ is a TS language for every $A \in N$.

Let $A \in N_i$, for some $i, 1 \leq i \leq 4$. Let $A \rightarrow \Omega_1 \mid \dots \mid \Omega_m$ be the rules for A in R . Define the language $L_{R_A} = \Omega_1 \cup \dots \cup \Omega_m \cup A^{+,+}$. Obviously, L_{R_A} is a TS language (on $N \cup \Sigma$), but it is also, by Definition 23, part (2), a $(N_i, \Sigma \cup (N - N_i))_i$ -corner language. Consider the grammar $G_A = (\Sigma \cup N - N_i, N_i, A, R)$, where the nonterminal alphabet is only N_i and the other symbols in N are considered as terminals. Define

for every j , $1 \leq j \leq 4$, the $(N_j, \Sigma \cup N - N_j)_j$ -corner substitutions $\sigma_j(X) = L_{R_X}$ for every $X \in N_j$, $\sigma_j(X) = X^{+,+}$ for $X \in \Sigma \cup N - N_j$. Then if $p \in \sigma_{i_C}^h(A^{+,+})$ for some $h > 0$ (i.e., it may be obtained by iterating h times the corner substitution σ_{i_C}) there is $k \leq h$ such that $A \rightrightarrows_{G_A}^k p$. Also, if $A \rightrightarrows_{G_A}^h p$ then $p \in \sigma_{i_C}^h(A^{+,+})$. Hence, $L(G_A) = \sigma_{i_C}^*(A^{+,+}) \cap (\Sigma \cup N - N_i)^{+,+}$, and hence $L(G_A)$ is a TS language since TS-REC is closed under intersection and nested iterated corner TS substitution. Define a substitution $\delta(X) = \square_{N-\bar{N}}(\sigma_{1_C}^*(\sigma_{2_C}^*(\sigma_{3_C}^*(\sigma_{4_C}^*(X^{+,+}))))$. Since, by definition of corner grammar, there is no mutual recursion between nonterminals in different alphabets N_i and N_j , there exists $k > 0$ such that $\delta_B^k(X^{+,+}) = \delta_B^{k+1}(X^{+,+})$. Since $\delta_B^k(X^{+,+})$ is a finite iteration of a TS substitution applied to a TS language, also $\delta_B^k(X^{+,+}) \cap (\Sigma \cup \bar{N})^{+,+}$ is TS language. For simplicity, define a block substitution $\rho : \bigcup_i N_i \cup \Sigma \rightarrow 2^{(\Sigma \cup \bar{N})^{+,+}}$ with $\rho(a) = a$ if $a \in \Sigma$, $\rho(A) = \delta_B^k(\square_{N-\bar{N}}(A^{+,+})) \cap (\Sigma \cup \bar{N})^{+,+}$ for $A \in \bigcup_i N_i$. Let $A \in \bar{N}$. Define a grammar $G'_A = (\Sigma \cup \bigcup_i N_i, \bar{N}, A, R)$. Grammar G'_A is not recursive and therefore derives a TS language $L(G'_A)$. Define a block substitution $\nu : \bar{N} \cup \Sigma \rightarrow 2^{(\Sigma \cup \bigcup_i N_i)^{+,+}}$ with $\nu(a) = a$ if $a \in \Sigma$, $\nu(A) = L(G'_A)$ for $A \in \bar{N}$. Finally, now one can compose ρ and ν to define the TS block substitution $\tau : \bar{N} \cup \Sigma \rightarrow 2^{(\Sigma \cup \bar{N})^{+,+}}$ as $\tau(a) = \rho(\nu(a^{+,+})) \cup a^{+,+}$ for every $a \in \bar{N} \cup \Sigma$. Then if $S \in \bar{N}$, $p \in L(G)$ iff there is j such that $p \in \tau^j(S^{+,+}) \cap \Sigma^{+,+}$, while if $S \in N_i$ then $p \in L(G)$ iff there is j such that $p \in \nu(\tau^j(\rho(S^{+,+}))) \cap \Sigma^{+,+}$. However, since from every $A \in \bar{N}$ it is not possible to generate a picture with a block A -subpicture (i.e., self-recursion is not possible for symbols in \bar{N}), then $j \leq |\bar{N}|$. Therefore, $L(G)$ is also a TS language, by closure under block substitution and intersection. \square

By Theorem 3, it follows immediately that for every TS language there is an equivalent corner grammar, since non-recursive TRG are just a special case of corner grammars.

Example 4. The corner grammar:

$$S \rightarrow B_{2,2} \left(\begin{array}{ccc} S & S & a \\ S & S & a \\ a & a & a \end{array} \right) \mid \begin{array}{c} S \ a \\ a \ a \end{array} \mid a$$

defines the language $\{a^{n,n} : n \geq 1\}$. The equivalent TS, conceptually based on the proof of Theorem 4, is defined by the tile set

$$\theta = B_{2,2}(\bar{p}_1) \cup B_{2,2}(\bar{p}_2); \quad p_1 = \begin{array}{ccc} S & S' & S \\ S' & S' & S \\ S & S & S \end{array}, \quad p_2 = \begin{array}{ccc} S & S' & S' \\ S' & S' & S' \\ S & S & S' \\ S' & S' & S' \end{array}$$

and by the projection $\pi(S) = \pi(S') = a$.

In conclusion, this paper went some way into showing that classical results of string languages can be extended to picture languages of the TRG and TS families: two properties related to recursion have been reformulated from context-free string grammars and proved valid in the 2D case.

Acknowledgements

We thank the anonymous referees for their critical suggestions, which helped in improving the paper.

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