

# SAT-TS: a SAT-based tool to recognize and complete pictures specified by tiling

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## Extended Abstract

Syntactic methods (see for instance [1],[2]) have been often considered for performing pattern analysis and recognition, by formally specifying the class of pictures of interest. Pictures or patterns can be specified by different methods, such as grammars or automata. A sample of approaches can be found in [3], including for instance [4], where isometric array grammars are considered for efficient syntactic pattern recognition and picture generation. An alternative, theoretically sound, yet practically unexplored, approach is to use tiling: in the crudest form a specified set of small, say two by two, tiles is listed, which can cover the intended class of pictures. A picture is recognized if, and only if, it can be covered with tiles from the listed set. To overcome the limitations of such rudimentary method, a more flexible formalism, called *Tiling Systems* (TS) has been studied by theoreticians (see e.g. [5], [6], [7]). *Wang Tiles* [8] are an equivalent variant of the formalism, which uses a more traditional concept of tiling where tiles are placed side by side. With TS the picture is obtained by first covering it with tiles drawn from a listed set of two by two tiles, then by performing a pixel by pixel mapping. Tiling Systems are a powerful technique: the corresponding pictures can be recognized by non-deterministic cellular automata, which orderly scan the diagonals [9]. Such abstract machines are more powerful than the four ways automata of [10]. However TS definitions are hard to write and error-prone for non elementary pictures. Moreover the NP-complete computational complexity of picture recognition has until now blocked any attempt to realistic experimentation and application of TS, in spite of a large amount of theoretical work.

Our work is concerned with a practical experimentation of tiling systems/Wang tiles in conjunction with a new approach for performing pattern recognition and image generation or completion, based on powerful logical tools, the SAT-solvers, whose task is to find Boolean values which make a propositional formula true. We have implemented a recognizer/generator for TS defined pictures in a very attractive, unconventional way, by transforming the tiling problem into a Boolean satisfiability one, then using an efficient off-the-shelf SAT-solver. The tool is invaluable to assist in writing picture specifications, is fast enough to experiment on reasonably sized samples, and has the bonus of being able to complete a partial picture, by assigning to unknown pixels some values which satisfy the picture specification. Therefore, SAT-TS can be also applied to image

reconstruction or noise elimination, by parsing a picture where some pixels are tagged as unknown.

A first, very simple and unoptimized prototype of the tool was presented at the ESF workshop *Advances on Two-dimensional language theory*, held in Salerno, Italy, May 3-5, 2006. We intend to show here the more solid and efficient new version, together with several examples and applications, such as the set of geographical maps which can be colored with three colors, and various classes of nested patterns and connected paths. SAT-TS is open source and freely available.<sup>1</sup>

### The Encoding

We briefly present here the main ideas and principles for encoding tiling systems into a SAT-problem. The actual encoding implemented in SAT-TS is a Conjunctive Normal Form optimized variant of the one presented next.

Consider a Tiling System  $T = (\Sigma, \Gamma, \Theta, \pi)$ , where  $\Sigma$  is the input alphabet,  $\Gamma$  is the local alphabet,  $\Theta$  is the set of tiles on  $\Gamma$ , and  $\pi : \Gamma \rightarrow \Sigma$  the projection.

Essentially, given an input picture  $p \in \Sigma^{*,*}$ , i.e. a picture made of symbols taken from  $\Sigma$ , the parsing problem consists in finding a picture  $q \in \Gamma^{*,*}$ , having the same size as  $p$ , such that:

1. its projection coincides with  $p$ , i.e.  $\pi(q) = p$ ;
2. its tiling is compatible with  $\Theta$ , i.e. every  $2 \times 2$  sub-picture of  $q$  is in  $\Theta$ .

If both conditions are true, then, and only then,  $p \in \mathcal{L}(T)$ . Clearly,  $q$  is not necessarily unique. Notice that this is an instance of typical inverse mathematical problems, which are often computationally challenging.

Now, to encode the problem into SAT, we represent the pixels of the picture  $q$  as SAT's propositional variables. In practice, this means that the statement  $q(i, j) = a$  (i.e. pixel  $(i, j)$  of  $q$  contains the symbol  $a$ ), becomes a propositional variable of the SAT problem.

To fully exploit the SAT encoding, we also accept partial input pictures. This means that some of  $p$ 's pixels may be left unspecified (conventionally marked by a “don't care” symbol “?”). With a slight abuse of notation, we say that the inverse projection of a “don't care” symbol in  $p$  is  $\Gamma$ , i.e.  $\pi^{-1}(?) = \Gamma$ . Informally, this means that we do not know anything about that pixel, so any symbol of the tile alphabet could be in  $q$  at that position.

The encoding consists of expressing the afore mentioned Conditions 1) and 2), as propositional logic formulas.

Condition 1) states that  $q$  must be “compatible” with  $p$ , i.e. such that  $\pi(q) = p$ :<sup>2</sup>

$$F_1 := \bigwedge_{(i,j) \in [(1,1) \dots |p|]} \left( \text{OnlyOne}_{a \in \pi^{-1}(p(i,j))} (q(i, j) = a) \right)$$

<sup>1</sup> <http://www.elet.polimi.it/upload/pradella/>

<sup>2</sup> For conciseness, we introduce the OnlyOne Boolean function, with any number of arguments. Informally, OnlyOne is true if, and only if, exactly one of its arguments is true. E.g.  $\text{OnlyOne}(A, B, C) \iff (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C)$ .

$F_1$  depends only on  $p$  and on the projection  $\pi$ . The first AND is used to span the whole picture, while the inner OnlyOne operator is used to check that one and only one value taken from the alphabet  $\Gamma$  is assigned to  $q$  at a given position.

Condition 2) considers the tile set  $\Theta$ : to accept  $p$ , every tile used in  $q$  must be a member of  $\Theta$ .

$$F_2 := \bigwedge_{(i,j) \in [(1,1)..|p|]} \bigvee_{t \in \Theta} \bigwedge_{h,k \in [0,1]} \left( \begin{array}{c} q(i+h, j+k) \\ = \\ t(h+1, k+1) \end{array} \right)$$

As in the previous formula, the first AND spans the whole picture. Then, the inner OR states that one of the tiles in  $\Theta$  must be present at a given position.

The TS-recognition problem is then encoded as the propositional formula  $F_1 \wedge F_2$ .

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