Cyclic Operator Precedence Grammars for Improved Parallel Parsing*

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Abstract. Operator precedence languages (OPL) enjoy the *local parsability property*, which essentially means that a code fragment enclosed within a pair of markers —playing the role of parentheses— can be compiled with no knowledge of its external context. Such a property has been exploited to build parallel compilers for languages formalized as OPLs. It has been observed, however, that when the syntax trees of the sentences have a linear substructure, its parsing must necessarily proceed sequentially making it impossible to split such a subtree into chunks to be processed in parallel. Such an inconvenience is due to the fact that so far much literature on OPLs has assumed the hypothesis that the equality precedence relation cannot be cyclic. We present an enriched version of operator precedence grammars which allows to remove the above hypothesis, therefore providing a little more expressive generality, and to further optimize parallel compilation.

Keywords: Operator Precedence Languages · Cyclic Precedence Relations · Parallel Parsing

1 Introduction

Operator precedence languages (OPL) are a "historical" family of languages invented by R. Floyd [10] to support fast deterministic parsing. Together with their *operator precedence grammars (OPG)*, they are still used within modern compilers to parse expressions with operators ranked by priority. The key feature that makes them well amenable for efficient parsing and compilation is that the syntax tree of a sentence is determined exclusively by three binary precedence relations over the terminal alphabet that are easily pre-computed from the grammar productions. For example: the arithmetic sentence $a + b \times c$ does not make manifest the natural structure $(a + (b \times c))$, but the latter is implied by the fact that the plus operator yields precedence to the times.

After Floyd's pioneering contribution much subsequent research discovered further important properties of OPLs, not necessarily related to deterministic parsing, that are

* The extended version of this paper, available in [6], contains an omitted correctness proof, explanatory examples and figures developed in more depth.

typical of the much less powerful family of regular languages and therefore enabled applications in fields such as automatic verification. In synthesis, they are: OPLs are a boolean algebra [8]; they are also closed under concatenation and Kleene's * [7]; they are characterized, besides original OPGs, by a special and simple family of pushdown automata, named *Operator Precedence automata (OPA)*, a *monadic second-order (MSO) logic* that naturally extends the classic one for regular languages [15], *operator precedence expressions (OPE)* which similarly extend traditional regular expressions [17], and in terms of a *syntactic congruence* with a finite number of equivalence classes (OPSC) [13]. Finally, an FO-complete temporal logic equivalent to *aperiodic OPLs* [17] has been defined which enabled the construction of a first model checker to verify properties of OPLs with a complexity comparable with that of model checkers for regular languages [5]. To the best of our knowledge, OPLs are the largest subclass of context-free languages that enjoys the same properties of regular ones.

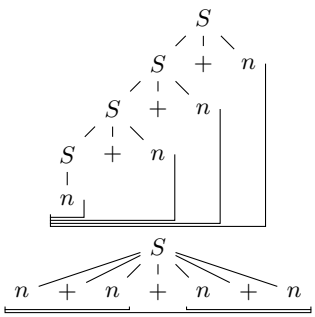


Fig. 1. Left-associative syntax tree (top) vs equal-level one (bottom) of the plus operator. The top syntax tree imposes a sequential left-to-right parsing and semantic processing whereas the bottom one can be split onto several branches to be partially processed independently and further aggregated.

It must be pointed out, however, that many —not all— of the algebraic properties discovered during such a research activity were proved by assuming a hypothesis on the precedence relations defined on the input alphabet. Although from a theoretical point of view this hypothesis slightly affects the generative power of OPGs —but is not necessary, e.g., for OPAs and MSO logic so that these two formalisms are a little more powerful than OPGs—, so far no practical limitation due to it was discovered in terms of formalizing the syntax of real-life programming and data description languages. Thus, it has been constantly adopted in the various developments to avoid making the mathematical notation and technical details too cumbersome.

Another distinguishing property of OPLs is their *local parsability*, i.e. the fact that a chunk included within a pair of symmetric precedence relations can be deterministically parsed even without knowing its context. This feature was exploited to produce a parallel parser generator which exhibited high performances w.r.t. traditional parsers [4,3].

The recent contribution [14], however, pointed out a weakness of the parallel compilation algorithm described in [3] which in some cases hampers partitioning the input to be parsed in well-balanced chunks, so that the benefits of parallelism are affected. Intuitively, the weakness is due to the fact that the "normal" precedence relation on the arithmetic operator + compels to parse a sequence thereof by associating them either to the left or to right so that parsing becomes necessarily sequential in this case. The authors also proposed a special technique to overtake this difficulty by allowing for an acceptable level of ambiguity, which in the case of OPGs determines a conflict for some precedence relations on the terminal alphabet.

Such normal precedence relation, however, which either lets + yield precedence to, or take precedence over itself, has no correspondence with the semantics of the operation,

whose result does not depend on the order of association. So, why not giving the various occurrences of the $+$ operator the same precedence level as suggested by arithmetic's laws? Fig. 1 gives an intuitive idea of the different structures given to a sequence of $+$ operators by traditional OPGs and the natural semantics of the sum operation.

The answer to this question comes exactly from the above mentioned hypothesis: it forbids cyclic sequences of symbols that are at the same level of precedence; so that $+$ cannot be at the same level of itself. Thus, the discovery of this practical restriction "compelled" us to finally remove this relatively disturbing hypothesis: this is the object of the present paper. The solution we devised consists in allowing grammar *right hand sides (rhs) to include the Kleene * operator*. Thus, we introduce the *Cyclic operator-precedence grammars (C-OPG)* which include the above feature: whereas such a feature is often used in general context-free grammars to make the syntax of programming languages more compact but does not increase their expressive power, we show that C-OPGs are now fully equivalent to OPAs and other formalisms to define OPLs, such as the MSO logic. We also show that all results previously obtained under the above hypothesis still hold by using C-OPGs instead of traditional OPGs. Although the goal of this paper is not to develop parallel compilation algorithms rooted in C-OPGs, we show how they naturally overtake the difficulty pointed out by [14] and would allow to revisit their techniques, or to improve the efficiency of our previous parallel parser [3].

2 Background

We assume some familiarity with the classical literature on formal language and automata theory, e.g., [19,12]. Here, we just list and explain our notations for the basic concepts we use from this theory. The terminal alphabet is usually denoted by Σ, and the empty string is ε. The character $\# \notin \Sigma$ is used as *delimiter*, and we define $\Sigma_\# = \Sigma \cup \{\#\}$.

A *context-free (CF) grammar* is a tuple $G = (\Sigma, V_N, P, S)$ where Σ and V_N, with $\Sigma \cap V_N = \emptyset$, are resp. the terminal and the nonterminal alphabets, the total alphabet is $V = \Sigma \cup V_N$, $P \subseteq V_N \times V^*$ is the rule (or production) set, and $S \subseteq V_N$, $S \neq \emptyset$, is the axiom set. For a generic rule, denoted as $A \to \alpha$, where A and α are resp. called the left/right hand sides (lhs / rhs), the following forms are relevant: *axiomatic* $A \in S$; *terminal* $\alpha \in \Sigma^+$; *empty* $\alpha = \varepsilon$; *renaming* $\alpha \in V_N$; *operator* $\alpha \notin V^*V_NV_NV^*$, i.e., at least one terminal is interposed between any two nonterminals occurring in α.

A grammar is *backward deterministic (BD)* if $(B \to \alpha, C \to \alpha \in P)$ implies $B = C$. If all rules of a grammar are in operator form, it is called an *operator grammar* or O-grammar. We give for granted the usual definition of *derivation* denoted by the symbols $\underset{G}{\Longrightarrow}$ (immediate derivation), $\overset{*}{\underset{G}{\Longrightarrow}}$ (reflexive and transitive closure of $\underset{G}{\Longrightarrow}$), $\overset{+}{\underset{G}{\Longrightarrow}}$ (transitive closure of $\underset{G}{\Longrightarrow}$); the subscript G will be omitted whenever clear from the context. We give also for granted the notion of *syntax tree (ST)*. As usual, the *frontier* of a syntax tree is the ordered left-to-right sequence of the leaves of the tree.

The *language* defined by G, said $L(G)$, is $\{w \mid w \in \Sigma^*, A \overset{*}{\underset{G}{\Longrightarrow}} w \wedge A \in S\}$. Two grammars defining the same language are *equivalent*. Two grammars generating the same set of syntax trees, up to a renaming of internal nodes, are *structurally equivalent*.

From now on, w.l.o.g., we exclusively deal with O-grammars without renaming and empty rules with the only exception that, if ε is part of the language, there is a unique empty rule whose lhs is an axiom that does not appear in the rhs of any production. In fact, this is a well-known normal form for CF grammars [2,12].

We now define operator precedence grammars (OPGs) following primarily [16]. Intuitively, OPGs are O-grammars whose parsing is driven by three *precedence relations*, called *equal*, *yield* and *take*, included in $\Sigma_\# \times \Sigma_\#$. They are defined in such a way that two consecutive terminals of a grammar's rhs —ignoring possible nonterminals in between— are in the equal relation, while the two extreme ones —again, whether or not preceded or followed by a nonterminal— are preceded by a yield and followed by a take relation, respectively; in this way a complete rhs of a grammar rule is identified and can be *reduced* to a corresponding lhs by a typical bottom-up parsing.

Definition 1 ([10]). *Let $G = (\Sigma, V_N, P, S)$ be an O-grammar. Let a, b denote elements in Σ, A, B in V_N, C either an element of V_N or the empty string ε, and α, β range over V^*. The left and right terminal sets of nonterminals are respectively:*
$\mathcal{L}_G(A) = \left\{ a \in \Sigma \mid \exists C : A \underset{G}{\overset{*}{\Longrightarrow}} Ca\alpha \right\}$ *and* $\mathcal{R}_G(A) = \left\{ a \in \Sigma \mid \exists C : A \underset{G}{\overset{*}{\Longrightarrow}} \alpha aC \right\}$.

The operator precedence (OP) relations *are defined over $\Sigma_\# \times \Sigma_\#$ as follows:*
Equal in precedence $a \doteq b \Leftrightarrow \exists A \to \alpha aCb\beta \in P$.
Takes precedence $a \gtrdot b \Leftrightarrow \exists A \to \alpha Bb\beta \in P, a \in \mathcal{R}(B); a \gtrdot \# \Leftrightarrow a \in \mathcal{R}(B), B \in S$.
Yields precedence $a \lessdot b \Leftrightarrow \exists A \to \alpha aB\beta \in P, b \in \mathcal{L}(B); \# \lessdot b \Leftrightarrow b \in \mathcal{L}(B), B \in S$.

The OP relations can be collected into a $|\Sigma_\#| \times |\Sigma_\#|$ array, called the operator precedence matrix *of the grammar, $OPM(G)$: for each (ordered) pair $(a, b) \in \Sigma_\# \times \Sigma_\#$, $OPM_{a,b}(G)$ contains the OP relations holding between a and b.*

An OPM is said conflict-free *iff $\forall a, b \in \Sigma_\#$, $0 \leq |M_{a,b}| \leq 1$. A conflict-free OPM is* total *or* complete *iff $\forall a, b \in \Sigma_\#$, $|M_{a,b}| = 1$. If $M_{\#,\#}$ is not empty, $M_{\#,\#} = \{\doteq\}$. An OPM is* $\doteq$-acyclic *if the transitive closure of the $\doteq$ relation over $\Sigma \times \Sigma$ is irreflexive.*

We extend the set inclusion relations and the Boolean operations in the obvious cell-by-cell way, to any two matrices having the same terminal alphabet. Two matrices are *compatible* iff their union is conflict-free.

Definition 2 (Operator precedence grammar). *A grammar G is an* operator precedence grammar *(OPG) iff the matrix $OPM(G)$ is conflict-free. An OPG is* $\doteq$-acyclic *if $OPM(G)$ is so. An* operator precedence language (OPL) *is a language generated by an OPG.*

Fig. 2 (left) displays an OPG, G_{AE}, which generates simple, unparenthesized arithmetic expressions and its OPM (center). The left and right terminal sets of G_{AE}'s nonterminals E, T and F are, respectively: $\mathcal{L}(E) = \{+, \times, n\}$, $\mathcal{L}(T) = \{\times, n\}$, $\mathcal{L}(F) = \{n\}$, $\mathcal{R}(E) = \{+, \times, n\}$, $\mathcal{R}(T) = \{\times, n\}$, and $\mathcal{R}(F) = \{n\}$.
Remark. If the relation $\doteq$ is acyclic, then the length of the rhs of any rule of G is bounded by the length of the longest $\doteq$-chain in $OPM(G)$.

The key feature of OPLs is that a conflict-free OPM M defines a universe of *strings compatible with* M and associates to each of them a unique *syntax tree* whose internal nodes are unlabeled and whose leaves are elements of Σ. We illustrate such a feature

through a simple example and refer the reader to previous literature for a thorough description of OP parsing [11,16].

$G_{AE}: S = \{E\}$
$E \to E + T \mid T \times F \mid n$
$T \to T \times F \mid n$
$F \to n$

	$+$	$\times$	n	$\#$
$+$	$\gtrdot$	$\lessdot$	$\lessdot$	$\gtrdot$
$\times$	$\gtrdot$	$\gtrdot$	$\lessdot$	$\gtrdot$
n	$\gtrdot$	$\gtrdot$		$\gtrdot$
$\#$	$\lessdot$	$\lessdot$	$\lessdot$	

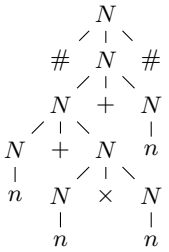


Fig. 2. G_{AE} (left), its OPM (center), and the syntax tree of $n + n \times n + n$ according to the OPM (right).

Example 3. Consider the $OPM(G_{AE})$ of Fig. 2 and the string $n + n \times n + n$. Display all precedence relations holding between consecutive terminal characters, *including the relations with the delimiters #* as shown here:

$$\# \lessdot n \gtrdot + \lessdot n \gtrdot \times \lessdot n \gtrdot + \lessdot n \gtrdot \#$$

Each pair $\lessdot, \gtrdot$ (with no further $\lessdot, \gtrdot$ in between) includes a *possible* rhs of a production of *any OPG* sharing the OPM with G_{AE}, not necessarily a G_{AE} rhs. Thus, as it happens in typical bottom-up parsing, we replace —*possibly in parallel*— each string included within the pair $\lessdot, \gtrdot$ with a *dummy nonterminal* N; this is because nonterminals are irrelevant for OPMs. The result is the string $\#N + N \times N + N\#$. Next, we compute again the precedence relation between consecutive terminal characters by *ignoring nonterminals*: the result is $\# \lessdot N + \lessdot N \times N \gtrdot + N \gtrdot \#$.

This time, there is only one pair $\lessdot, \gtrdot$ including a potential rhs determined by the OPM (the fact that the external $\lessdot$ and $\gtrdot$ "look matched" is coincidental as it can be easily verified by repeating the previous procedure with the string $n + n \times n + n + n$). Again, we replace the pattern $N \times N$, with the dummy nonterminal N; notice that there is no doubt about associating the two N to the $\times$ rather than to one of the adjacent $+$ symbols: if we replaced, say, just the $\times$ with an N we would obtain the string $N + NNN + N$ which cannot be derived by an O-grammar. By recomputing the precedence relations we obtain the string $\# \lessdot N + N \gtrdot + N \gtrdot \#$. Finally, by applying twice the replacing of $N + N$ by N we obtain $\#N\#$.

The result of the whole bottom-up reduction procedure is synthetically represented by the *syntax tree* of Fig. 2 (right) which shows the precedence of the multiplication operation over the additive one in traditional arithmetics. It also suggests a natural association to the left of both operations: if we reverted the order of the rhs of the rules rewriting E and T, the structure of the tree would have suggested associativity to the right of both operations which would not have altered the semantics of the two operations which can indifferently be associated to the left and to the right; not so is we dealt with, say, subtraction or division which instead *impose* association to the left.

Notice that the tree of Fig. 2 has been obtained —uniquely and deterministically— by using exclusively the OPM, not the grammar G_{AE} although the string $n+n\times n+n \in L(G_{AE})$ [4].

Obviously, all sentences of $L(G_{AE})$ can be given a syntax tree by $OPM(G_{AE})$, but there are also strings in Σ^* that can be parsed according to the same OPM but are not in $L(G_{AE})$. E.g., the string $+++$ is parsed according to the $OPM(G_{AE})$ as a ST that associates the $+$ characters to the left. Notice also that, in general, not every string in Σ^* is assigned a syntax tree by an OPM; e.g., in the case of $OPM(G_{AE})$ the parsing procedure applied to nn is immediately blocked since there is no precedence relation between n and itself.

Definition 4 (OP-alphabet and Maxlanguage). *A string in Σ^* is* compatible *with an OPM M iff the procedure described in Example 3 terminates by producing the pattern $\#N\#$. The set of all strings compatible with an OPM M is called the* maxlanguage *or the* universe *of M and is simply denoted as $L(M)$.*

Let M be a conflict-free OPM over $\Sigma_\# \times \Sigma_\#$. We use the same identifier M to denote the —partial— function M that assigns to strings in Σ^* their unique ST as informally illustrated in Example 3.

The pair (Σ, M) where M is a conflict-free OPM over $\Sigma_\# \times \Sigma_\#$, is called an OP-alphabet*. We introduce the concept of OP-alphabet as a pair to emphasize that it defines a universe of strings on the alphabet Σ —not necessarily covering the whole Σ^*— and implicitly assigns them a structure univocally determined by the OPM, or, equivalently, by the function M. The class of (Σ, M)-*compatible *OPGs and OPLs are respectively: $\mathscr{G}_M = \{G \mid G \text{ is an OPG and } OPM(G) \subseteq M\}$, $\mathscr{L}_M = \{L(G) \mid G \in \mathscr{G}_M\}$.*

Various formal properties of OPGs and OPLs are documented in the literature, e.g., in [8,7,16]. The next proposition recalls those that are relevant for this article.

Proposition 5 (Algebraic properties of OPGs and OPLs). *If an OPM M is total, then the corresponding homonymous function is total as well, i.e., $L(M) = \Sigma^*$.*

Let (Σ, M) be an OP-alphabet where M is $\doteq$-acyclic. The class $\mathscr{G}_M$ contains an OPG, called the maxgrammar *of M, denoted by $G_{max,M}$, which generates the maxlanguage $L(M)$. For all grammars $G \in \mathscr{G}_M$, $L(G) \subseteq L(M)$.*

The closure properties of the family $\mathscr{L}_M$ of (Σ, M)-compatible OPLs defined by a total OPM are the following:

- *$\mathscr{L}_M$ is closed under union, intersection and set-difference, therefore also under complement (if a maxgrammar of M exists).*
- *$\mathscr{L}_M$ is closed under concatenation; if M is $\doteq$-acyclic, $\mathscr{L}_M$ is closed under *.*

In terms of expressive power, OPLs are strictly in-between Visibly Pushdown Languages [1] and deterministic context-free languages.

[4] The above procedure that led to the syntax tree of Fig. 2 could be easily adapted to become an algorithm that produces a new syntax tree whose internal nodes are labeled by G_{AE}'s nonterminals. Such an algorithm could be made deterministic by transforming G_{AE} into a structurally equivalent BD grammar sharing the same OPM [2,12].

Remark. Thanks to the fact that a conflict-free OPM assigns to each string at most one ST —and exactly one if the OPM is complete— the above closure properties of OPLs w.r.t. Boolean operations automatically extend to sets of their STs. The same does not apply to the case of concatenation which in general may produce significant reshaping of the original STs [7]. Furthermore, any complete, conflict-free, $\doteq$-acyclic OPM defines a *universe of STs* whose frontiers are Σ^*.

The notion of *chain* introduced next is an alternative way to represent STs where internal nodes are irrelevant and "anonymized".

Definition 6 (Chains). *Let (Σ, M) be an OP-alphabet. A* simple chain *is a word $a_0a_1a_2\dots\ a_na_{n+1}$, written as ${}^{a_0}[a_1a_2\dots a_n]^{a_{n+1}}$, such that: $a_0, a_{n+1} \in \Sigma \cup \{\#\}$, $a_i \in \Sigma$ for every $i : 1 \le i \le n$, $M_{a_0a_{n+1}} \neq \emptyset$, and $a_0 \lessdot a_1 \doteq a_2 \dots a_{n-1} \doteq a_n \gtrdot a_{n+1}$.*

A composed chain *is a word $a_0x_0a_1x_1a_2\dots a_nx_na_{n+1}$, with $x_i \in \Sigma^*$, where ${}^{a_0}[a_1a_2\dots a_n]^{a_{n+1}}$ is a simple chain, and either $x_i = \varepsilon$ or ${}^{a_i}[x_i]^{a_{i+1}}$ is a chain (simple or composed), for every $i : 0 \le i \le n$. Such a composed chain will be written as ${}^{a_0}[x_0a_1x_1a_2\dots a_nx_n]^{a_{n+1}}$.*

The body of a chain ${}^a[x]^b$, simple or composed, is the word x. Given a chain ${}^a[x]^b$ the depth *$d(x)$ of its body x is defined recursively: $d(x) = 1$ if the chain is simple, whereas $d(x_0a_1x_1\dots a_nx_n) = 1 + \max_i d(x_i)$. The depth of a chain is the depth of its body.*

For instance, the ST of Fig. 2 (right) is biunivocally represented by the composed chain ${}^{\#}[x_0 + x_1]^{\#}$, where, in turn x_0 is the body of the composed chain ${}^{\#}[y_0 + y_1]^{+}$, y_0 is the body of the simple chain ${}^{\#}[n]^{+}$, y_1 is the body of the composed chain ${}^{+}[z_0 \times z_1]^{+}$, etc. The depth of the main chain is 3.

As well as an OPG selects a set of STs within the universe defined by its OPM, an *operator precedence automaton (OPA)* selects a set of chains within the universe defined by an OP-alphabet.

Definition 7 (Operator precedence automaton (OPA)). *A nondeterministic* OPA *is given by a tuple: $\mathcal{A} = \langle \Sigma, M, Q, I, F, \delta \rangle$ where: (Σ, M) is an operator precedence alphabet, Q is a set of states (disjoint from Σ), $I \subseteq Q$ is a set of initial states, $F \subseteq Q$ is a set of final states, δ, the transition function, is a triple of functions $\delta_{shift} : Q \times \Sigma \to \wp(Q), \delta_{push} : Q \times \Sigma \to \wp(Q), \delta_{pop} : Q \times Q \to \wp(Q)$.*

We represent a nondeterministic OPA by a graph with Q as the set of vertices and $\Sigma \cup Q$ as the set of edge labelings. We write $p \xrightarrow{a} q$ iff $q \in \delta_{\text{push}}(p, a)$, $p \overset{a}{\dashrightarrow} q$ iff $q \in \delta_{\text{shift}}(p, a)$, and $q \xRightarrow{p} r$ iff $r \in \delta_{\text{pop}}(q, p)$.

To define the semantics of the automaton, we introduce some notations. We use letters $p, q, p_i, q_i, \dots$ to denote states in Q. Let Γ be $\Sigma \times Q$ and let Γ' be $\Gamma \cup \{\bot\}$; we denote symbols in Γ' as $[a,\ q]$ or $\bot$. We set $symbol([a,\ q]) = a$, $symbol(\bot) = \#$, and $state([a,\ q]) = q$. Given a string $\Pi = \bot\pi_1\pi_2\dots\pi_n$, with $\pi_i \in \Gamma$, $n \ge 0$, we set $symbol(\Pi) = symbol(\pi_n)$, including the particular case $symbol(\bot) = \#$.

A *configuration* of an OPA is a triple $C = \langle \Pi,\ q,\ w \rangle$, where $\Pi \in \bot\Gamma^*$, $q \in Q$ and $w \in \Sigma^*\#$. The first component represents the contents of the stack, the second component represents the current state of the automaton, while the third component is the part of input still to be read.

A *computation* or *run* of the automaton is a finite sequence of *moves* or *transitions* $C_1 \vdash C_2$; there are three kinds of moves, depending on the precedence relation between the symbol on top of the stack and the next symbol to read:

push move: if $symbol(\Pi) \lessdot a$ then $\langle \Pi,\ p,\ ax\rangle \vdash \langle \Pi[a,\ p],\ q,\ x\rangle$, with $q \in \delta_{\text{push}}(p, a)$;

shift move: if $a \doteq b$ then $\langle \Pi[a,\ p],\ q,\ bx\rangle \vdash \langle \Pi[b,\ p],\ r,\ x\rangle$, with $r \in \delta_{\text{shift}}(q, b)$;

pop move: if $a \gtrdot b$ then $\langle \Pi[a,\ p],\ q,\ bx\rangle \vdash \langle \Pi,\ r,\ bx\rangle$, with $r \in \delta_{\text{pop}}(q, p)$.

Shift and pop moves are never performed when the stack contains only $\bot$.

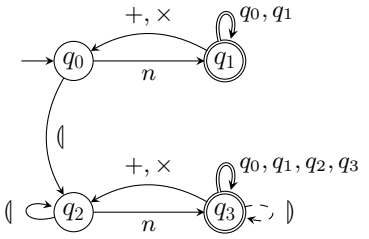
Push and shift moves update the current state of the automaton according to the transition functions δ_{push} and δ_{shift}, respectively: push moves put a new element on the top of the stack consisting of the input symbol together with the current state of the automaton, whereas shift moves update the top element of the stack by changing its input symbol only. The pop move removes the symbol on the top of the stack, and the state of the automaton is updated by δ_{pop} on the basis of the pair of states consisting of the current state of the automaton and the state of the removed stack symbol; notice that in this move the input symbol is used only to establish the $\gtrdot$ relation and it remains available for the following move.

A configuration $\langle \bot,\ q_I,\ x\#\rangle$ is *initial* if $q_I \in I$; a configuration $\langle \bot,\ q_F,\ \#\rangle$ is *accepting* if $q_F \in F$. The language accepted by the automaton is:
$L(\mathcal{A}) = \{x \mid \langle \bot,\ q_I,\ x\#\rangle \vdash^* \langle \bot,\ q_F,\ \#\rangle, q_I \in I, q_F \in F\}$.

Example 8. The OPA depicted in Fig. 3 (top, left) based on the OPM at the (top, right) accepts the language of arithmetic expressions enriched w.r.t $L(G_{AE})$ in that it introduces the use of explicit parentheses to alter the natural precedence of arithmetic operations. The same figure (bottom) also shows an accepting computation on input $n+n\times(\!(n+n)\!)$.

Definition 9. *Let $\mathcal{A}$ be an OPA. A* support *for a simple chain ${}^{a_0}[a_1 a_2 \dots a_n]^{a_{n+1}}$ is any path in $\mathcal{A}$ of the form $q_0 \xrightarrow{a_1} q_1 \dashrightarrow \dots \dashrightarrow q_{n-1} \overset{a_n}{\dashrightarrow} q_n \overset{q_0}{\Longrightarrow} q_{n+1}$.*
Notice that the label of the last (and only) pop is exactly q_0, i.e. the first state of the path; this support is built due to the relations $a_0 \lessdot a_1$ and $a_n \gtrdot a_{n+1}$.
A support for the composed chain ${}^{a_0}[x_0 a_1 x_1 a_2 \dots a_n x_n]^{a_{n+1}}$ *is any path in $\mathcal{A}$ of the form $q_0 \overset{x_0}{\rightsquigarrow} q'_0 \xrightarrow{a_1} q_1 \overset{x_1}{\rightsquigarrow} q'_1 \overset{a_2}{\dashrightarrow} \dots \overset{a_n}{\dashrightarrow} q_n \overset{x_n}{\rightsquigarrow} q'_n \overset{q'_0}{\Longrightarrow} q_{n+1}$, where for every $i, 0 \le i \le n$: if $x_i \neq \varepsilon$, then $q_i \overset{x_i}{\rightsquigarrow} q'_i$ is a support for the (simple or composed) chain ${}^{a_i}[x_i]^{a_{i+1}}$; if $x_i = \varepsilon$, then $q'_i = q_i$. Notice that the label of the last pop is exactly q'_0. The support of a chain with body x will be denoted by $q_0 \overset{x}{\rightsquigarrow} q_{n+1}$.*

The context a, b of a chain ${}^a[x]^b$ is used by the automaton to build its support only because $a \lessdot x$ and $x \gtrdot b$; thus, the chain's body contains all information needed by the automaton to build the subtree whose frontier is that string, once it is understood that its first move is a push and its last one is pop. This is a distinguishing feature of OPLs, not shared by other deterministic languages: we call it their *locality principle*, which has been exploited to build parallel and/or incremental OP parsers [4].



	$+$	$\times$	(	)	n	#
$+$	$\gtrdot$	$\lessdot$	$\lessdot$	$\gtrdot$	$\lessdot$	$\gtrdot$
$\times$	$\gtrdot$	$\gtrdot$	$\lessdot$	$\gtrdot$	$\lessdot$	$\gtrdot$
(	$\lessdot$	$\lessdot$	$\lessdot$	$\doteq$	$\lessdot$	
)	$\gtrdot$	$\gtrdot$		$\gtrdot$		$\gtrdot$
n	$\gtrdot$	$\gtrdot$		$\gtrdot$		$\gtrdot$
#	$\lessdot$	$\lessdot$	$\lessdot$		$\lessdot$	

stack	state	current input
$\bot$	q_0	$n + n \times (n + n)\#$
$\bot[n,\ q_0]$	q_1	$+n \times (n + n)\#$
$\bot$	q_1	$+n \times (n + n)\#$
$\bot[+,\ q_1]$	q_0	$n \times (n + n)\#$
$\bot[+,\ q_1][n,\ q_0]$	q_1	$\times(n + n)\#$
$\bot[+,\ q_1]$	q_1	$\times(n + n)\#$
$\bot[+,\ q_1][\times,\ q_1]$	q_0	$(n + n)\#$
$\bot[+,\ q_1][\times,\ q_1][(,\ q_0]$	q_2	$n + n)\#$
$\bot[+,\ q_1][\times,\ q_1][(,\ q_0][n,\ q_2]$	q_3	$+n)\#$
$\bot[+,\ q_1][\times,\ q_1][(,\ q_0]$	q_3	$+n)\#$
$\bot[+,\ q_1][\times,\ q_1][(,\ q_0][+,\ q_3]$	q_2	$n)\#$
$\bot[+,\ q_1][\times,\ q_1][(,\ q_0][+,\ q_3][n,\ q_2]$	q_3	$)\#$
$\bot[+,\ q_1][\times,\ q_1][(,\ q_0][+,\ q_3]$	q_3	$)\#$
$\bot[+,\ q_1][\times,\ q_1][(,\ q_0]$	q_3	$)\#$
$\bot[+,\ q_1][\times,\ q_1][),\ q_0]$	q_3	$\#$
$\bot[+,\ q_1][\times,\ q_1]$	q_3	$\#$
$\bot[+,\ q_1]$	q_3	$\#$
$\bot$	q_3	$\#$

Fig. 3. An OPA (top, left), its OPM (top, right) and an example of computation for the language of Example 8 (bottom). Arrows $\longrightarrow$, $\dashrightarrow$ and $\Longrightarrow$ denote push, shift and pop transitions, respectively. To avoid confusion with the overloaded parenthesis symbols, the parentheses used as terminal symbols are denoted as (and).

3 Cyclic Operator Precedence Grammars (C-OPGs)

Proposition 5 shows that *some, but not all*, of the algebraic properties of OPLs depend critically on the $\doteq$-acyclicity hypothesis. This is due to the fact that without such a hypothesis the rhs of an OPG have an unbounded length but cannot be infinite: e.g., no OPG can generate the language $\{a, b\}^*$ if $a \doteq b$ and $b \doteq a$. In most cases cycles of this type can be "broken" as it has been done up to now, e.g., to avoid the $+ \doteq +$ relation in arithmetic expressions by associating the operator indifferently to the right or to left. From a theoretical point of view, the $\doteq$-acyclicity hypothesis affects the expressive power of OPGs; thus, the OPL familily as generated by OPGs is strictly included within the languages accepted by OPAs.[5] We assumed so far the $\doteq$-acyclicity hypothesis to keep the notation as simple as possible so that the two formalisms are equivalent.

Recently, however, it has been observed [14] that such a restriction may hamper the benefits achievable by the parallel compilation techniques that exploit the local

[5] The language $\{a^n(bc)^n\} \cup \{b^n(ca)^n\} \cup \{c^n(ab)^n\} \cup (abc)^+$ cannot be generated by an OPG because the $a \doteq b \doteq c \doteq a$ relations are necessary [9], but it is accepted by OPAs.

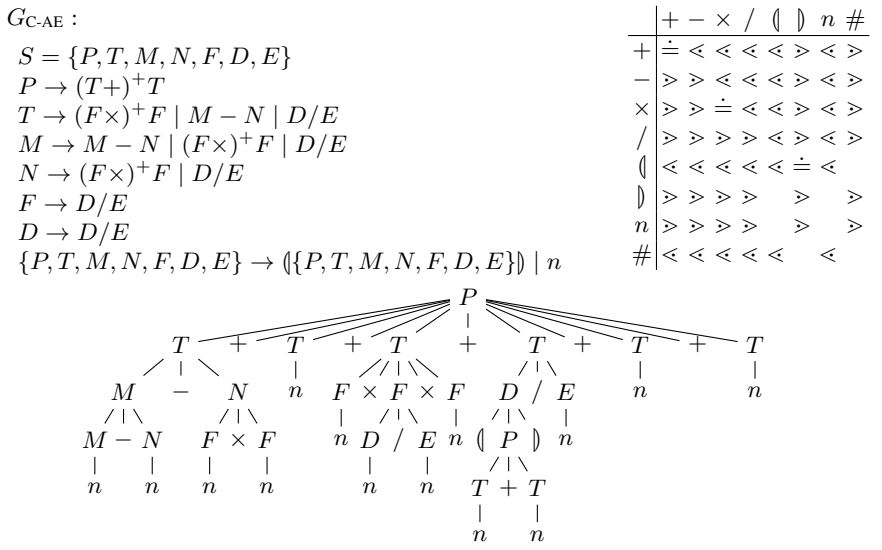


Fig. 4. A C-OPG (top left), its OPM (top right), and a ST generated by them. The notation $\{P, T, M, F, D, E\} \to (\!|\{P, T, M, N, F, D, E\}|\!)$ means that anyone of the nonterminals at the left can be rewritten as a pair of parentheses enclosing anyone of the same nonterminals.

parsability property of OPLs [3]. Thus, it is time to introduce the necessary extension of OPGs so that the $\doteq$-acyclicity hypothesis can be avoided and they become fully equivalent to other formalisms to define OPLs.

Definition 10 (Cyclic Operator Precedence Grammar (C-OPG)). *A $^+$-O-expression on V^* is an expression obtained from the elements of V by iterative application of concatenation and the $^+$ operator [6], provided that any substring thereof has no two adjacent nonterminals; for convenience, and w.l.o.g., we assume that all subexpressions that are argument of the $^+$ operator are terminated by a terminal character.*

A Cyclic O-grammar (C-OG) is an O-grammar whose production rhs are $^+$-O-expressions. For a rule $A \to \alpha$ of a C-OG, the $\underset{G}{\Longrightarrow}$ (immediate derivation) relation is defined as $\beta A\gamma \Longrightarrow \beta\zeta\gamma$ iff ζ is a string belonging to the language defined by the $^+$-O-expression α, $L(\alpha)$. The $\doteq$ relation is redefined as $a \doteq b$ iff $\exists A \to \alpha \wedge \exists \zeta = \eta aCb\theta \mid (C \in V_N \cup \{\varepsilon\} \wedge \zeta \in L(\alpha))$. The other relations remain defined as for non-cyclic O-grammars. A C-OG is a C-OPG iff its OPM is conflict-free.

As a consequence of the definition of the immediate derivation relation for C-OPGs the STs derived therefrom can be unranked, i.e., their internal nodes may have an unbounded number of children.

[6] For our purposes $^+$ is more convenient than * without affecting the generality.

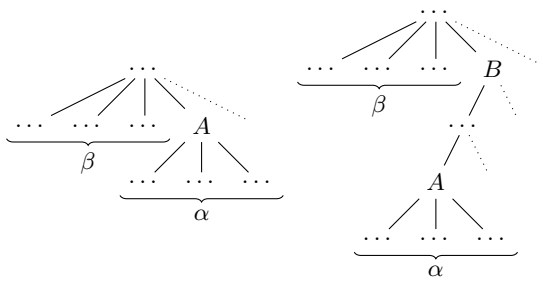


Fig. 5. When parsing α, the prefix previously under construction is β.

Example 11. The C-OPG shown in Fig. 4 with its OPM generates a fairly complete language of parenthesized arithmetic expressions involving the four basic operations: as usual the multiplicative operations take precedence over the additive ones; subtraction takes precedence over sum and division over multiplication. The key novelty w.r.t. the traditional way of formalizing arithmetic expressions by means of OPGs are the $+ \doteq +$ and $\times \doteq \times$ OP relations; on the contrary we kept the structure that associates subtraction and division to the left, so that the grammar's STs —an example thereof is given at the bottom of Fig. 4— now fully reflect the semantics of arithmetic operations.

By looking at the ST of Fig. 4 and comparing it with the original Fig. 1, one can better envision why the introduction of cyclic $\doteq$ can support more effective parallel parsing algorithms for OPLs. Parallel parsing for OPLs is rooted in the *local parsability property* of this family: thanks to this property any fragment of input string enclosed within a pair of corresponding $\lessdot$ and $\gtrdot$ OP relations can be processed in parallel with other similar fragments. However, if, say, the $+$ operator is associated to the left (or right) as in the case of the upper ST of Fig. 1 the parsing of a sequence of $+$ must necessarily proceed sequentially from left to right. Conversely, if the ST has a structure like that of the lower tree of Fig. 1 the sequence of $+$ —whether intermixed or not with other subtrees— can be arbitrarily split into several branches which can be parsed in parallel and, after that, can be joined into a unique subtree as imposed by the $\doteq$ OP relation between the corresponding extreme terminals of contiguous branches.

3.1 Equivalence between C-OPGs and OPAs

The equivalence is obtained by adapting the analogous proof given in [15] where the additional hypothesis of M being $\doteq$-acyclic was exploited. First, we describe a procedure to build an OPA equivalent to a C-OPG. Then, we provide the converse construction.

Theorem 12 (From C-OPGs to OPAs). *Let (Σ, M) be an OP-alphabet. For any C-OPG defined thereon an equivalent OPA can be effectively built.*

A nondeterministic OPA[7] $\mathcal{A} = \langle \Sigma, M, Q, I, F, \delta \rangle$ from a given C-OPG G with the same precedence matrix M as G is built in such a way that a successful computation

[7] Any nondeterministic OPA can be transformed into a deterministic one at the cost of quadratic exponential increase in the size of the state space [15].

thereof corresponds to building bottom-up a syntax tree of G: the automaton performs a push transition when it reads the first terminal of a new rhs; it performs a shift transition when it reads a terminal symbol inside a rhs, i.e. a leaf with some left sibling leaf. It performs a pop transition when it completes the recognition of a rhs, then it guesses (nondeterministically) the nonterminal at the lhs. Each state contains two pieces of information: the first component is the prefix of the rhs under construction, whereas the second component is used to recover the rhs *previously under construction* whenever all rhs nested below have been completed (see Fig. 5).

Let $\hat{P}$ be the set of rhs γ where all $^+$ and related parentheses have been erased. Let $\tilde{P}$ be the set of strings $\tilde{\gamma} \in V^+$ belonging to the language of some rhs γ of P that is inductively defined as follows: if $(\eta)^+$ is a subexpression of γ such that η is a single string $\in V^+$ then $\tilde{\eta} = \{\eta, \eta\eta\}$; if $\eta = \alpha_1(\beta_1)^+\alpha_2(\beta_2)^+ \dots \alpha_n$ where $\alpha_i \in V^*$, then $\tilde{\eta} = \{\eta_1, \eta_1\eta_1\}$ where $\eta_1 = \alpha_1\tilde{\beta}_1\alpha_2\tilde{\beta}_2 \dots \alpha_n$.
E.g., let η be $(Ba(bc)^+)^+$; then $\hat{\eta} = \{Babc\}$ and $\tilde{\eta} = \{Babc, Babcbc, BabcBabc, BabcbcBabc, BabcBabcbc, BabcbcBabcbc\}$.

Let $\mathbb{P} = \{\alpha \in V^*\Sigma \mid \exists A \to \eta \in P \wedge \exists\beta(\alpha\beta \in \tilde{\eta})\}$ be the set of prefixes, ending with a terminal symbol, of strings $\in \tilde{P}$; define $\mathbb{Q} = \{\varepsilon\} \cup \mathbb{P} \cup N$, $Q = \mathbb{Q} \times (\{\varepsilon\} \cup \mathbb{P})$, $I = \{\langle\varepsilon, \varepsilon\rangle\}$, and $F = S \times \{\varepsilon\} \cup \{\langle\varepsilon, \varepsilon\rangle$ if $\varepsilon \in L(G)\}$. Note that $|\mathbb{Q}| = 1 + |\mathbb{P}| + |N|$ is $O(m^h)$ where m is the maximum length of the rhs in P, and h is the maximum nesting level of $^+$ operators in rhs; therefore $|Q|$ is $O(m^{2h})$.

The transition functions are defined by the following formulas, for $a \in \Sigma$ and $\alpha, \alpha_1, \alpha_2 \in \mathbb{Q}$, $\beta, \beta_1, \beta_2 \in \{\varepsilon\} \cup \mathbb{P}$, and where for any expression ξ, $\bar{\xi}$ is obtained from ξ by erasing parentheses and $^+$ operators:

- $\delta_{\text{shift}}(\langle\alpha, \beta\rangle, a) \ni \begin{cases} \text{if } \alpha \notin N: \begin{cases} \text{if } \begin{pmatrix} \exists A \to \gamma \mid \gamma = \eta(\zeta)^+\theta \wedge \\ \alpha a = \bar{\eta}\bar{\zeta}\bar{\zeta} \wedge \alpha a\bar{\theta} \in L(\gamma) \cap \tilde{P} \end{pmatrix} \\ \text{then } \langle\bar{\eta}\bar{\zeta}, \beta\rangle \text{ else } \langle\alpha a, \beta\rangle \end{cases} \\ \text{if } \alpha \in N: \begin{cases} \text{if } \begin{pmatrix} \exists A \to \gamma \mid \gamma = \eta(\zeta)^+\theta \wedge \\ \beta\alpha a = \bar{\eta}\bar{\zeta}\bar{\zeta} \wedge \beta\alpha a\bar{\theta} \in L(\gamma) \cap \tilde{P} \end{pmatrix} \\ \text{then } \langle\bar{\eta}\bar{\zeta}, \beta\rangle \text{ else } \langle\beta\alpha a, \beta\rangle \end{cases} \end{cases}$
- $\delta_{\text{push}}(\langle\alpha, \beta\rangle, a) \ni \begin{cases} \langle a, \alpha\rangle & \text{if } \alpha \notin N \\ \langle\alpha a, \beta\rangle & \text{if } \alpha \in N \end{cases}$
- $\delta_{\text{pop}}(\langle\alpha_1, \beta_1\rangle, \langle\alpha_2, \beta_2\rangle) \ni \langle A, \gamma\rangle$
  $\forall A: \begin{cases} \text{if } \alpha_1 \notin N: A \to \alpha \in P \wedge \alpha_1 \in L(\alpha) \cap \hat{P} \\ \text{if } \alpha_1 \in N: A \to \delta \in P \wedge \beta_1\alpha_1 \in L(\delta) \cap \hat{P} \end{cases}$ and $\gamma = \begin{cases} \alpha_2 \text{ if } \alpha_2 \notin N \\ \beta_2 \text{ if } \alpha_2 \in N. \end{cases}$

The states reached by push and shift transitions have the first component in $\mathbb{P}$. If state $\langle\alpha, \beta\rangle$ is reached after a push transition, then α is the prefix of the rhs (deprived of the $^+$ operators) that is currently under construction and β is the prefix previously under construction; in this case α is either a terminal or a nonterminal followed by a terminal.

If the state is reached after a shift transition, and the α component of the previous state was not a single nonterminal, then the new α is the concatenation of the first component of the previous state with the read character. If, instead, the α component of the previous state was a single nonterminal —which was produced by a pop transition— then the new α also includes the previous β and β is not changed from the previous state. However, if the new α becomes such that a suffix thereof is a double occurrence of a

stack	state	current input
$\bot$	$\langle \varepsilon, \varepsilon \rangle$	$n+n+n/n/n+n+n\#$
$\bot[n,\ \langle \varepsilon, \varepsilon \rangle]$	$\langle n, \varepsilon \rangle$	$+n+n/n/n+n+n\#$
$\bot$	$\langle T, \varepsilon \rangle$	$+n+n/n/n+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle]$	$\langle T+, \varepsilon \rangle$	$n+n/n/n+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle][n,\ \langle T+, \varepsilon \rangle]$	$\langle n, T+ \rangle$	$+n/n/n+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle]$	$\langle T, T+ \rangle$	$+n/n/n+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle]$	$\langle \mathbf{T}+, \mathbf{T}+ \rangle$	$n/n/n+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle][n,\ \langle T+, T+ \rangle]$	$\langle n, T+ \rangle$	$/n/n+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle]$	$\langle D, T+ \rangle$	$/n/n+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle][/,\ \langle D, T+ \rangle]$	$\langle D/, T+ \rangle$	$n/n+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle][/,\ \langle D, T+ \rangle][n,\ \langle /, D \rangle]$	$\langle n, D/ \rangle$	$/n+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle][/,\ \langle D, T+ \rangle]$	$\langle E, D/ \rangle$	$/n+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle]$	$\langle D, T+ \rangle$	$/n+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle][/,\ \langle D, T+ \rangle]$	$\langle D/, T+ \rangle$	$n+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle][/,\ \langle D, T+ \rangle][n,\ \langle D/, T+ \rangle]$	$\langle n, D/ \rangle$	$+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle][/,\ \langle D, T+ \rangle]$	$\langle E, D/ \rangle$	$+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle]$	$\langle T, T+ \rangle$	$+n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle]$	$\langle \mathbf{T}+, \mathbf{T}+ \rangle$	$n+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle][n,\ \langle T+, T+ \rangle]$	$\langle n, T+ \rangle$	$+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle]$	$\langle T, T+ \rangle$	$+n\#$
$\bot[+,\ \langle T, \varepsilon \rangle]$	$\langle \mathbf{T}+, \mathbf{T}+ \rangle$	$n\#$
$\bot[+,\ \langle T, \varepsilon \rangle][n,\ \langle T+, T+ \rangle]$	$\langle n, T+ \rangle$	$\#$
$\bot[+,\ \langle T, \varepsilon \rangle]$	$\langle T, T+ \rangle$	$\#$
$\bot$	$\langle P, \varepsilon \rangle$	$\#$

Fig. 6. A run of the OPA built from the C-OPG of Fig. 4 accepting the sentence $n+n+n/n/n+n+n$. The states truncated by erasing a repeated suffix ζ occurring under the scope of a $^+$ operator are emphasized in boldface.

string $\zeta \in L((\zeta)^+)$ —hence $\alpha \in \mathbb{P}$— then the second occurrence of ζ is cut from the new α, which therefore becomes a prefix of an element of $\hat{P}$.

The states reached by a pop transition have the first component in N: if $\langle A, \gamma \rangle$ is such a state, then A is the corresponding lhs, and γ is the prefix previously under construction.

For instance, imagine that a C-OPG contains the rules $A \to (Ba(bc)^+)^+a$ and $B \to h$ and that the corresponding OPA $\mathcal{A}$ parses the string $habcbchabca$: after scanning the prefix $habcb$ $\mathcal{A}$ has reduced h to B and has $Babcb$ as the first component of its state; after reading the new c it recognizes that the suffix of the first state component would become a second instance of bc belonging to $(bc)^+$; thus, it goes back to $Babc$. Then, it proceeds with a new reduction of h to B and, when reading with a shift the second a appends Ba to its current β which was produced by the previous pop so that the new α becomes $BabcBa$; after shifting b it reads c and realizes that its new α would become $BabcBabc$, i.e., an element of $(Ba(bc)^+)^+$ and therefore "cuts" it to the single instance thereof, i.e., $Babc$. Finally, after having shifted the last a it is ready for the last pop.

The result of δ_{shift} and δ_{push} is a singleton, whereas δ_{pop} may produce several states, in case of repeated rhs. Thus, if G is BD, the corresponding $\mathcal{A}$ is deterministic.

Example 13. Fig. 6 displays a run of the OPA obtained from the C-OPG of Fig. 4 accepting the sentence $n+n+n/n/n+n+n$.

The construction of a C-OPG equivalent to a given OPA is far simpler than the converse one, thanks to the explicit structure associated to words by the precedence

matrix. The key difference w.r.t. the analogous construction given in [15] is that even simple chains can have unbounded length because the $\doteq$ relation may be circular.

In analogy with the definition of $\tilde{P}$, we define *essential supports* as those supports where possible cyclic behaviors of the OPA along a sequence of terminals —whether with interposing nonterminals or not— occur exactly twice.

Definition 14. *An essential support of a simple chain* ${}^{a_0}[a_1 a_2 \dots a_n]^{a_{n+1}}$ *is any path in* $\mathcal{A}$ *of the form* $q_0 \xrightarrow{a_1} q_1 \dashrightarrow \dots \dashrightarrow q_{n-1} \overset{a_n}{\dashrightarrow} q_n \overset{q_0}{\Longrightarrow} q_{n+1}$*, where any cycle* $q_i a_{i1} \dots a_{ik} q_i$ *is repeated exactly twice. Composed chains are treated similarly.*

E.g., with reference to the OPA built from the C-OPG of Fig. 4 an essential support of the chain ${}^{\#}[n+n+n+n+n]^{\#}$ is: $\langle \varepsilon, \varepsilon \rangle \overset{n}{\rightsquigarrow} \langle T, \varepsilon \rangle \xrightarrow{+} \langle T+, \varepsilon \rangle \overset{n}{\rightsquigarrow} \langle T, T+ \rangle \overset{+}{\dashrightarrow} \langle T+, T+ \rangle \overset{n}{\rightsquigarrow} \langle T, T+ \rangle \overset{+}{\dashrightarrow} \langle T+, T+ \rangle \overset{n}{\rightsquigarrow} \langle T, T+ \rangle \overset{\langle T, \varepsilon \rangle}{\Longrightarrow} \langle P, \varepsilon \rangle$.

Lemma 15. *The essential supports of simple chains of any OPA have an effectively computable bounded length.*

Theorem 16 (From OPAs to C-OPGs). *Let* (Σ, M) *be an OP-alphabet. For any OPA defined thereon an equivalent C-OPG can be effectively built.*

Proof. Given an OPA $\mathcal{A} = \langle \Sigma, M, Q, I, F, \delta \rangle$, we build an equivalent C-OPG G having OPM M. The equivalence between $\mathcal{A}$ and G is then rather obvious.

G's nonterminals are the 4-tuples $(a, q, p, b) \in \Sigma \times Q \times Q \times \Sigma$, written as $\langle {}^{a}p, q^{b} \rangle$. G's rules are built as follows:

For every essential support of a simple chain, P contains $\langle {}^{a_0}q_0, {q_{n+1}}^{a_{n+1}} \rangle \to a_1 a_2 \dots a_n$, where every double sequence $a_{i1} \dots a_{ik} a_{i1} \dots a_{ik}$ is recursively replaced by $(a_{i1} \dots a_{ik})^+$ by proceeding from the innermost cycles to the outermost ones. Furthermore, if $a_0 = a_{n+1} = \#$, q_0 is initial, and q_{n+1} is final, then $\langle {}^{\#}q_0, {q_{n+1}}^{\#} \rangle$ is in S.

For every essential support of a composed chain ${}^{a_0}[x_0 a_1 x_1 a_2 \dots a_n x_n]^{a_{n+1}}$, P contains the rule $\langle {}^{a_0}q_0, {q_{n+1}}^{a_{n+1}} \rangle \to \Lambda_0 a_1 \Lambda_1 a_2 \dots a_n \Lambda_n$, where, for every $i = 0, 1, \dots, n$, $\Lambda_i = \langle {}^{a_i}q_i, {q'_i}^{a_{i+1}} \rangle$ if $x_i \neq \varepsilon$ and $\Lambda_i = \varepsilon$ otherwise, and double cyclic sequences $\alpha_i \alpha_i$ are replaced by $(\alpha_i)^+$ in the same way as for simple chains. If $a_0 = a_{n+1} = \#$, q_0 is initial, and q_{n+1} is final, then $\langle {}^{\#}q_0, {q_{n+1}}^{\#} \rangle$ is in S; if ε is accepted by $\mathcal{A}$, $A \to \varepsilon$ is in P, A being a new axiom not otherwise occurring in any other rule.

Notice that the above construction is effective thanks to Lemma 15 and to the fact that subchains of composed chains are replaced by nonterminals Λ_i.

4 Equivalences and Closure Properties

Previous literature proved the equivalence of defining OPLs through OPGs, OPAs, MSO logic, OPEs and OPSC. The reciprocal inclusions between MSO and OPA, between OPA and the finite equivalence classes of OPL syntactic congruence, and the inclusion of MSO in OPE have been proved without the hypothesis of non-circularity of the $\doteq$ relation; the reciprocal inclusions between C-OPG and OPA have been restated in Section 3; it remains the inclusion of OPE in OPG which was proved in [17] under the restrictive hypothesis. The proof used the claim that deriving an OPG from an OPE may

exploit the closure properties of OPLs, in particular, w.r.t. *; such a closure, however, was proved in [7] by using OPGs, again, under the restrictive hypothesis.

In OPEs the * operator is applied to subexpressions independently on the OP relations between their last and first terminal character. Thus, it is first convenient to rewrite the OPE in a normal form using the $^+$ operator instead of the * one to avoid having to deal explicitly with the case of the ε string. Then, subexpressions of type $(\alpha)^+$ where the last terminal of α is not in relation $\doteq$ with the first one are replaced by the same procedure defined in [7] to prove effectively the closure w.r.t. the * operator. The new rules will produce a right or left-linear subtree of the occurrences of α depending on the OP relation between the two extreme terminals of α and will avoid the use of the * and $^+$ operators which are not permitted in the original OPGs.

The remaining substrings including the $^+$ operator are the new rhs of the C-OPG. The other technicalities of the construction of an OPG equivalent to an OPE are identical to those given in [17] and are not repeated here.

All major closure properties of OPLs have been originally proved by referring to their generating OPGs and some of them, in particular the closure w.r.t. *, required the $\doteq$-acyclicity hypothesis. Thus, it is necessary to prove them again. However, since some of those proofs are technically rather involved, here we simply observe that it is easier to restate the same properties by exploiting OPAs which are now fully equivalent to C-OPGs. Thanks to the determinization of nondeterministic OPAs, closure w.r.t. boolean operations is "for free". Closure w.r.t. concatenation can be seen as a corollary of the closure proved in [15] of the concatenation between an OPL whose strings have finite length and an ω-OPL, i.e., a language of infinite strings. The construction is based on a nondeterministic guess of the position where a string of the first language could end —and a nontrivial technique to decide whether it could be accepted even in the absence of the $\#$ delimiter—. Then, the closure w.r.t. * is obtained simply by allowing the OPA to repeat such a guess any number of times until the real $\#$ is found.

5 Conclusion

We have filled up a longstanding "hole" in the theory of OPLs under the pressure of recent applications in the field of parallel compilation that showed how such a hole could hamper the benefits of parallelism [14]. The new formalism of C-OPGs, fully equivalent to OPAs, MSO-logic, and OPEs, can be exploited for the parallel compilation techniques of [14] or to improve the efficiency of techniques based on the less powerful OPGs [3].

Other algebraic and logic properties of OPLs, e.g., aperiodicity, star-freeness, first-order definability [18,17] can be re-investigated in the light of this generalization.

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